A Generalized Homogeneous and Self-Dual Algorithm for Linear Programming

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Abstract: A generalized homogeneous and self-dual (HSD) infeasible-interior-point algorithm for linear programming (LP) is proposed in this paper. The algorithm does not need to start from a big M initial point, while achieving $O(\frac{\sqrt{n}}{\beta(1-\beta)}L)$-iteration complexity by following a certain central path on a central surface in a neighborhood $\mathcal{N}(\beta)$, where β can be any number between 0 and 1, n is the number of variables and L is the data length of the LP problem. In particular, an algorithm is developed, where the searching direction is obtained by solving a Newton equation system without infeasible residual terms on its right hand side.

Key words: Linear programming, homogeneous and self-dual linear feasibility model, interior-point algorithm

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1 Introduction

Consider a linear programming (LP) problem in the standard form:

$$\begin{array}{lll} (LP) & \text{minimize} & c^T x \\ & \text{subject to} & Ax = b, \ x \geq 0, \end{array}$$

where $c \in R^n$, $A \in R^{m \times n}$ and $b \in R^m$ are given, $x \in R^n$, and ^{T} denotes transpose. (LP) is said to be feasible if and only if its constraints are consistent; it is called unbounded if there is a sequence $\{x^k\}$ such that x^k is feasible for all k but $c^T x^k \to -\infty$. (LP) has a solution if and only if it is feasible and bounded.

The dual problem of (LP) can be written as

$$\begin{array}{lll} (LD) & \text{maximize} & b^T y \\ & \text{subject to} & A^T y \leq c, \end{array}$$

where $y \in R^m$. We call $z = c - A^T y \in R^n$ dual slacks. Denote by $\mathcal{F}$ the set of all x and (y, z) that are feasible for the primal and dual, respectively. Denote by $\mathcal{F}^0$ the set of points with $(x, z) > 0$ in $\mathcal{F}$.

Assuming that the LP problem has a feasible interior point, Megiddo [8] and Bayer and Lagarias [1] defined the central path for a feasible LP problem as

$$\mathcal{C}(\mu) = \left\{ (y, x, z) \in \mathcal{F}^0 : Xz = \mu e, \quad \mu = \frac{x^T z}{n} \right\}$$

where $X = diag(x)$. As $\mu \to 0$, this path goes to a strictly complementary solution of LP. Based on following the central path, Kojima et al. [4] developed a primal-dual interior-point algorithm in which the searching direction is generated by solving the following Newton equation system in iteration k:

$$\begin{array}{rrrcl} & A\, d_x & & = & 0, \\ -A^T\, d_y & & -d_z & = & 0, \\ & Z^k d_x & +X^k d_z & = & \gamma \mu^k e - X^k z^k, \end{array}$$

where $\mu^k = ((x^k)^T z^k)/n$, $X^k = diag(x^k)$, $Z^k = diag(z^k)$, and γ is a scalar parameter.

Kojima et al. [4] proved that their algorithm is $O(nL)$-iteration bounded, where L is the data length of (LP) with integer numbers. Later, Kojima et al. [5] and Monteiro and Adler [12] gave an $O(\sqrt{n}L)$-iteration bound for such a primal-dual interior-point algorithm by restricting all iterates in a 2-norm neighborhood of the central path, i.e.

$$(y^k, x^k, z^k) \in \mathcal{N}(\beta) = \left\{ (y, x, z) \in \mathcal{F}^0 : \|Xz - \mu e\| \leq \beta\mu \right\}$$

for some $\beta \in (0, 1/2]$. (Typically, $\beta = 1/4$, or for a predictor-corrector algorithm $\beta = 1/2$ in the predictor step, see [11].) Throughout the paper, $\|.\|$ represents 2-norm.

Unless a LP problem has a feasible interior point and such a point is given, an interior-point algorithm has to start from an infeasible point, or from an interior point feasible for an artificial problem. In theory, a big M interior point suffices for establishing complexity result. However, such a big M approach is not practical at all. Furthermore, a robust algorithm has to be able to correct possible error accumulated from computations even starting from a feasible interior point. Algorithms, which allow to start from a non-big M initial point in both theory and practice, are called infeasible interior-point algorithms and reported perform very well in practice (see [6], [7], [9], [10], [14], [16], and [19]). Unlike for feasible algorithms (in which a feasible interior point is given as the initial point), the best-to-date $O(\sqrt{n}L)$-iteration complexity for infeasible interior-point algorithms was not established until Ye et al. [18] proposed an homogeneous and self-dual (HSD) algorithm.

Recently Mizuno et al. [11] studied the trajectories followed by many primal-dual infeasible interior-point algorithms. For given $(y^0, x^0 > 0, z^0 > 0)$, they defined two dimensional central surface $\{\mathcal{Q}(\theta,\mu) : \quad \theta \geq \prime,\ \mu \geq \prime\}$ with

$$\mathcal{Q}(\theta,\mu) = \left\{ (y, x > 0, z > 0) : Xz = \mu e, \quad \mu = \frac{x^T z}{n}, \quad \begin{pmatrix} r_P \\ r_D \end{pmatrix} = \theta \begin{pmatrix} r_P^0 \\ r_D^0 \end{pmatrix} \right\}$$

where

$$r_P^0 = b - Ax^0 \quad and \quad r_D^0 = c - A^T y^0 - z^0, \qquad r_P = b - Ax \quad and \quad r_D = c - A^T y^k - z^k$$

are primal and dual residuals respectively. If the LP problem possesses a solution, many primal-dual infeasible interior-point algorithms (e.g., Kojima et al. [6], Lustig [7], Mehrotra [9]) follow some paths on this central surface and approach optimality and feasibility simultaneously:

$$\begin{aligned} for\ t \to 0 : \quad & \mu(t) \to 0, \\ & \theta(t) \to 0. \end{aligned}$$

Mizuno et al. [11] also discussed in detail the boundary behavior of central surface for primal-dual type infeasible interior-point algorithms.

Very recently, Xu et al. [17] proposed a simplified version of the HSD Algorithm of Ye et al. [18]. The algorithm deals with a homogeneous and self-dual linear feasibility model

$$\begin{array}{lcccl} (HLF) & & Ax & -b\tau & = 0, \\ & -A^T y & & +c\tau & \geq 0, \\ & b^T y & -c^T x & & \geq 0, \\ & y \text{ free} & x \geq 0 & \tau \geq 0. & \end{array} \tag{1}$$

Denote by z the slack vector for the second (inequality) constraint and by κ the slack scalar for the third (inequality) constraint. Then, the problem is to find a strictly complementary point such that

$$x^T z = 0 \quad \text{and} \quad \tau\kappa = 0.$$

The kth iteration of the HSD algorithm solves the following system of linear equations for direction $(d_y, d_x, d_\tau, d_z, d_\kappa)$

$$\begin{array}{rrrrrcl} & A\ d_x & -b\ d_\tau & & & = & \eta\ r_P^k, \\ -A^T\ d_y & & +c\ d_\tau & -d_z & & = & -\eta\ r_D^k, \\ b^T\ d_y & -c^T\ d_x & & & -d_\kappa & = & \eta\ r_G^k, \end{array} \tag{2}$$

and

$$\begin{array}{rcl} X^k d_z + Z^k d_x & = & \gamma\mu^k e - X^k z^k, \\ \tau^k d_\kappa + \kappa^k d_\tau & = & \gamma\mu^k - \tau^k \kappa^k, \end{array} \tag{3}$$

where $\gamma > 0$, η are scalar parameters, and

$$\mu^k = ((x^k)^T z^k + \tau^k \kappa^k)/(n+1), \tag{4}$$

$$r_P^k = b\tau^k - Ax^k, \quad r_D^k = c\tau^k - A^T y^k - z^k, \quad r_G^k = c^T x^k - b^T y^k + \kappa^k. \tag{5}$$

Xu et al. [17] showed that if we set $\eta = 1 - \gamma$ in each iteration, then the algorithm becomes the HSD algorithm of Ye et al. [18], which follows a path $\{\mathcal{Q}(\mu^0 t, \theta^0 t) : \ 0 \le t \le 1\}$ on the central surface. More precisely, Xu et al. [17] set

$$\Omega(\frac{\mu^0}{\theta^0}) \le \frac{\mu^k}{\theta^k} \le O(\frac{\mu^0}{\theta^0}).$$

The limit points of these paths are strictly complementary point for (HLF), according to Mizuno et al. [11]. If setting $\eta < 1 - \gamma$ or $\eta > 1 - \gamma$ at each iteration, then the algorithm generates iterates converging to the all-zero solution or diverging, respectively.

In this paper by introducing a simple update, we generalize the HSD algorithm of Xu et al. [17] so that a strictly complementary solution is obtained even when $\eta \ne 1 - \gamma$. In section 3, we prove that the generalized algorithm achieves $O(\frac{\sqrt{n}}{\beta(1-\beta)}L)$-iteration complexity by following certain central path on the central surface in a neighborhood $\mathcal{N}(\beta)$, where β can be any number $\in (0, 1)$. By setting $\eta = 0$, we get an interesting algorithm in which the searching direction is obtained by solving a Newton equation system without infeasible residual terms on its right hand side, as first proposed by de Ghellinck and Vial [2] and later by Nesterov [13]. This approach obviously saves the computation of these residual terms.

2 Generalized HSD algorithms

Generic HSD algorithm

Given initial point y^0, $x^0 > 0, \tau^0 > 0, z^0 > 0, \kappa^0 > 0$, $k \leftarrow 0$.

While (stopping criteria not satisfied) do

1. Let $r_P^k = b\tau^k - Ax^k$, $\quad r_D^k = c\tau^k - A^T y^k - z^k$, $\quad r_G^k = c^T x^k - b^T y^k + \kappa^k$.

2. Solve (2) and (3) for $d_y, d_x, d_\tau, d_z, d_\kappa$.

3. Let

$$
\begin{aligned}
\delta_x &= d_x + (1-\gamma-\eta)x^k, \\
\delta_y &= d_y + (1-\gamma-\eta)y^k, \\
\delta_z &= d_z + (1-\gamma-\eta)z^k, \\
\delta_\tau &= d_\tau + (1-\gamma-\eta)\tau^k, \\
\delta_\kappa &= d_\kappa + (1-\gamma-\eta)\kappa^k.
\end{aligned} \tag{6}
$$

4. Choose a step size $\alpha^k > 0$ and update

$$
\begin{aligned}
x^{k+1} &= x^k + \alpha^k \delta_x > 0, \\
y^{k+1} &= y^k + \alpha^k \delta_y, \\
z^{k+1} &= z^k + \alpha^k \delta_z > 0, \\
\tau^{k+1} &= \tau^k + \alpha^k \delta_\tau > 0, \\
\kappa^{k+1} &= \kappa^k + \alpha^k \delta_\kappa > 0.
\end{aligned} \tag{7}
$$

5. $k \leftarrow k+1$.

Note that we have

$$
\begin{aligned}
X^k \delta_z + Z^k \delta_x &= X^k d_z + Z^k d_x + (1-\gamma-\eta)(X^k z^k + Z^k x^k) \\
&= \gamma\mu^k e - X^k z^k + 2(1-\gamma-\eta)X^k z^k \\
&= \gamma\mu^k e - (2\gamma+2\eta-1)X^k z^k, \\
\kappa^k \delta_\tau + \tau^k \delta_\kappa &= \gamma\mu^k - (2\gamma+2\eta-1)\tau^k \kappa^k.
\end{aligned} \tag{8}
$$

Similar to the proofs in Xu et al. [17], we first establish the following lemmas:

Lemma 1 . *The direction resulting from (6) satisfies*

$$
(\delta_x)^T \delta_z + \delta_\tau \delta_\kappa = 0. \tag{9}
$$

Proof. Xu et al. [17] established following result for the solution of system (2) and (3):

$$
d_x^T d_z + d_\tau d_\kappa = \eta(1-\gamma-\eta)(n+1)\mu.
$$

Thus

$$
\begin{aligned}
(\delta_x)^T \delta_z + \delta_\tau \delta_\kappa &= d_x^T d_z + d_\tau d_\kappa + (1-\gamma-\eta)((x^k)^T d_z + (z^k)^T d_x + \tau^k d_\kappa + \kappa^k d_\tau) + \\
&\quad (1-\gamma-\eta)^2((x^k)^T z^k + \tau^k \kappa^k) \\
&= [\eta(1-\gamma-\eta) + (1-\gamma-\eta)(\gamma-1) + (1-\gamma-\eta)^2](n+1)\mu \\
&= 0.
\end{aligned}
$$

Q.E.D.

Lemma 2 . *The generic algorithm generates* $\{\mu^k\}$ *and* $\{\theta^k\}$ *satisfying*

$$\mu^0 = [(x^0)^T z^0 + \tau^0 \kappa^0] \ / \ (n+1), \quad \mu^{k+1} = (1 + \alpha^k(1 - \gamma - 2\eta))\mu^k \tag{10}$$

and

$$\theta^0 = 1, \quad \theta^{k+1} = (1 + \alpha^k(1 - \gamma - 2\eta))\theta^k \tag{11}$$

such that

$$r_P^k = \theta^k r_P^0, \quad r_D^k = \theta^k r_D^0, \quad r_G^k = \theta^k r_G^0.$$

Proof. By (9) and (8), we have

$$\begin{aligned}
\mu^{k+1} &= [(x^{k+1})^T z^{k+1} + (\tau^{k+1})^T \kappa^{k+1}] \ / \ (n+1) \\
&= \{ \ [(x^k)^T z^k + \tau^k \kappa^k] + \ \alpha^k[(x^k)^T \delta_z + (\delta_x)^T z^k + \tau^k \delta_\kappa + \delta_\tau \kappa^k] \ \} \ / \ (n+1) \\
&= [1 + \alpha^k(\gamma - 2\gamma - 2\eta + 1)] \ \mu^k \\
&= [1 + \alpha^k(1 - \gamma - 2\eta)] \ \mu^k.
\end{aligned}$$

By (2), (6) and (7), we also have

$$\begin{aligned}
r_P^{k+1} &= [(\tau^k + \alpha^k \delta_\tau)b - A(x^k + \alpha^k \delta_x)] \\
&= [r_P^k + \alpha^k(\delta_\tau b - A\delta_x)] \\
&= [1 + \alpha^k(1 - \gamma - 2\eta)] r_P^k.
\end{aligned}$$

Similarly, we have this relation for r_D^{k+1} and r_G^{k+1} as well.

Q.E.D.

From Lemma 2, for any choice of η and γ, our algorithm ensures μ^k and θ^k having a nice fixed ratio

$$\mu^k/\theta^k = \mu^0/\theta^0.$$

The non-negativity of $(x^{k+1}, \tau^{k+1}, z^{k+1}, \kappa^{k+1})$ results in $\mu^{k+1} \geq 0$, which implies that the step size α^k must satisfy

$$1 + \alpha^k(1 - \gamma - 2\eta) \geq 0.$$

Therefore, letting

$$\gamma + 2\eta - 1 > 0, \tag{12}$$

yields

$$0 \leq 1 + \alpha^k(1 - \gamma - 2\eta) < 1.$$

According to Mizuno et al. [11], we have the following corollary.

Corollary 3 . *If the generic algorithm generates* $\{(y^k, x^k, \tau^k, z^k, \kappa^k)\}$ *satisfying*

$$\theta^k \to 0$$

and

$$\min[\min_i(x_i^k z_i^k), \tau^k \kappa^k] \geq \beta\mu^k$$

for certain $\beta > 0$*, then every limit point of the sequence is a strictly complementary solution of (HLF).*

3 $O(\frac{\sqrt{n}}{\beta(1-\beta)}L)$-iteration HSD algorithms

For (HLF) the two dimensional central surface and its neighborhood are defined as

$$\mathcal{Q}(\theta,\mu) = \left\{ (y,\ x>0, \tau>0, z>0, \kappa>0) : \begin{pmatrix} Xz \\ \tau\kappa \end{pmatrix} = \mu e, \begin{pmatrix} r_P \\ r_D \\ r_G \end{pmatrix} = \theta \begin{pmatrix} r_P^0 \\ r_D^0 \\ r_G^0 \end{pmatrix} \right\}$$

$$\mathcal{N}(\beta) = \left\{ (y,\ x>0, \tau>0, z>0, \kappa>0) : \left\| \begin{pmatrix} Xz \\ \tau\kappa \end{pmatrix} - \mu e \right\| \leq \beta\mu, \begin{pmatrix} r_P \\ r_D \\ r_G \end{pmatrix} = \theta \begin{pmatrix} r_P^0 \\ r_D^0 \\ r_G^0 \end{pmatrix} \right\}$$

for some $\beta \in (0,1)$, respectively.

Theorem 4 . *For a given* $0 < \beta < 1$ *and* $(y^k, x^k, \tau^k, z^k, \kappa^k) \in \mathcal{N}(\beta)$*, if*

$$\alpha^k \leq \min\{\frac{1}{2\gamma+2\eta-1}\ ,\ \frac{2(1-\beta)\gamma}{\beta[(\gamma+2\eta-1)\sqrt{n+1}/\beta + (2\gamma+2\eta-1)]^2}\} \tag{13}$$

then

$$\left\| \begin{pmatrix} X^{k+1}z^{k+1} \\ \tau^{k+1}\kappa^{k+1} \end{pmatrix} - \mu^{k+1}e \right\| \leq \beta\mu^{k+1}.$$

Proof. To simplify the notation, we use x and z to represent $\begin{pmatrix} x \\ \tau \end{pmatrix}$ and $\begin{pmatrix} z \\ \kappa \end{pmatrix}$. Therefore $\|e\| = \sqrt{n+1}$. Note that this notation is only employed in the proof of this theorem. As usual, the capital expression denotes the diagonal matrix of a vector. Thus $\Delta_x = diag(\delta_x)$, $\Delta_z = diag(\delta_z)$.

Consider

$$\begin{aligned} & \|X^{k+1}z^{k+1} - \mu^{k+1}e\| \\ = \ & \|X^k z^k + \alpha^k(X^k\delta_z + Z^k\delta_x) + (\alpha^k)^2\Delta_x\delta_z - [1+\alpha^k(1-\gamma-2\eta)]\mu^k e\| \\ \leq \ & \|X^k z^k + \alpha^k(X^k\delta_z + Z^k\delta_x) - [1+\alpha^k(1-\gamma-2\eta)]\mu^k e\| + (\alpha^k)^2\|\Delta_x\delta_z\| \\ = \ & |1-\alpha^k(2\gamma+2\eta-1)|\ \|X^k z^k - \mu^k e\| + (\alpha^k)^2\|\Delta_x\delta_z\|. \end{aligned}$$

Using $\delta_x^T \delta_z = 0$, we have

$$\begin{aligned}
& ||\Delta_x \delta_z|| \\
= \;& ||[(X^k)^{-1} Z^k]^{1/2} \Delta_x \; [X^k (Z^k)^{-1}]^{1/2} \Delta_z e|| \\
\leq \;& \tfrac{1}{2} ||[(X^k)^{-1} Z^k]^{1/2} \Delta_x e \; + \; [X^k (Z^k)^{-1}]^{1/2} \Delta_z e||^2 \\
= \;& \tfrac{1}{2} ||(X^k Z^k)^{-1/2} (Z^k \delta_x \; + \; X^k \delta_z)||^2 \\
\leq \;& \tfrac{1}{2 \min_i x_i z_i} ||Z^k \delta_x \; + \; X^k \delta_z||^2 \\
= \;& \tfrac{1}{2 \min_i x_i z_i} ||\gamma \mu^k e - (2\gamma + 2\eta - 1) X^k z^k||^2 \\
\leq \;& \tfrac{1}{2 \min_i x_i z_i} [||(\gamma + 2\eta - 1)\mu^k e|| + ||(2\gamma + 2\eta - 1)(X^k z^k - \mu^k e)||]^2 \\
\leq \;& \tfrac{(\mu^k)^2}{2 \min_i x_i^k z_i^k} [(\gamma + 2\eta - 1)\sqrt{n+1} + (2\gamma + 2\eta - 1)\beta]^2.
\end{aligned}$$

By $\min_i x_i^k z_i^k \geq (1 - \beta)\mu^k$, we have

$$\begin{aligned}
& ||X^{k+1} z^{k+1} - \mu^{k+1} e|| \\
\leq \;& |1 - \alpha^k (2\gamma + 2\eta - 1)| \; \beta \mu^k + \tfrac{(\alpha^k)^2 (\mu^k)^2}{2 \min_i x_i^k z_i^k} [(\gamma + 2\eta - 1)\sqrt{n+1} + (2\gamma + 2\eta - 1)\beta]^2 \\
= \;& \{|1 - \alpha^k (2\gamma + 2\eta - 1)| \; + \tfrac{\beta (\alpha^k)^2}{2(1-\beta)} [(\gamma + 2\eta - 1)\sqrt{n+1}/\beta + (2\gamma + 2\eta - 1)]^2\} \beta \mu^k.
\end{aligned}$$

Again, Lemma 2 tells us

$$\mu^{k+1} = [1 - \alpha^k (\gamma + 2\eta - 1)] \mu^k. \tag{14}$$

Therefore,

$$||X^{k+1} z^{k+1} - \mu^{k+1} e|| \leq \beta \mu^{k+1}$$

if

$$|1 - \alpha^k (2\gamma + 2\eta - 1)| \; + \frac{\beta (\alpha^k)^2}{2(1 - \beta)} [(\gamma + 2\eta - 1)\sqrt{n+1}/\beta + (2\gamma + 2\eta - 1)]^2 \leq [1 - \alpha^k (\gamma + 2\eta - 1)],$$

or

$$\frac{\beta}{2(1 - \beta)} [(\gamma + 2\eta - 1)\sqrt{n+1}/\beta + (2\gamma + 2\eta - 1)]^2 (\alpha^k)^2 \leq [1 - \alpha^k (\gamma + 2\eta - 1)] - |1 - \alpha^k (2\gamma + 2\eta - 1)|.$$

If we further assume

$$1 - \alpha^k (2\gamma + 2\eta - 1) \geq 0,$$

then it becomes

$$\alpha^k \leq \frac{2(1 - \beta)\gamma}{\beta [(\gamma + 2\eta - 1)\sqrt{n+1}/\beta + (2\gamma + 2\eta - 1)]^2}.$$

Thus we have proved the theorem.

Q.E.D.

Using the simple continuarity argument ([11]), we see from Theorem 4 that, as long as the step size α^k satisfies (13), the resulting point is still in the neighborhood of the central path $(y^{k+1}, x^{k+1}, \tau^{k+1}, z^{k+1}, \kappa^{k+1}) \in \mathcal{N}(\beta)$.

Let us now consider the following optimization problem for a given $0 < \beta < 1$

$$\begin{array}{ll} \text{minimize} & \mu^{k+1}/\mu^k \\ \text{subject to} & \alpha^k, \gamma, \eta \quad \text{satisfy} \quad (12), (13). \end{array} \tag{15}$$

Setting step size according to (13), we obtain

$$\begin{aligned} \mu^{k+1}/\mu^k &= 1 \ - \ \alpha^k(\gamma + 2\eta - 1) \\ &= 1 \ - \ \min\{\tfrac{1}{2\gamma+2\eta-1} \ , \ \tfrac{2(1-\beta)\gamma}{\beta[(\gamma+2\eta-1)\sqrt{n+1}/\beta+(2\gamma+2\eta-1)]^2}\} \ (\gamma + 2\eta - 1). \end{aligned}$$

Letting $\omega = \gamma + 2\eta - 1$, we can rewrite problem (15) as $(0 < \beta < 1)$

$$\begin{array}{ll} \text{minimize} & 1 \ - \ \min\{\frac{1}{\gamma+\omega} \ , \ \frac{2(1-\beta)\gamma}{\beta(\omega\sqrt{n+1}/\beta+\gamma+\omega)^2}\} \ \omega \\ \text{subject to} & \omega > 0, \quad \gamma > 0. \end{array}$$

By setting $\bar{\omega} = \frac{\omega}{\gamma}$, problem (15) further becomes

$$\begin{array}{ll} \text{minimize} & 1 \ - \ \min\{\frac{\bar{\omega}}{1+\bar{\omega}} \ , \ \frac{2(1-\beta)\bar{\omega}}{\beta[(\sqrt{n+1}/\beta+1)\bar{\omega}+1]^2}\} \\ \text{subject to} & \bar{\omega} > 0. \end{array} \tag{16}$$

It is easy to verify that the problem

$$\text{minimize} \quad 1 \ - \ \frac{2(1-\beta)\bar{\omega}}{\beta[(\sqrt{n+1}/\beta+1)\bar{\omega}+1]^2}$$

has the optimal value

$$1 - \frac{1-\beta}{2(\sqrt{n+1}+\beta)}$$

with the optimizer

$$\bar{\omega}^* = \frac{1}{\sqrt{n+1}/\beta + 1}.$$

This implies that γ and η satisfy

$$\gamma = (\sqrt{n+1}/\beta + 1)(\gamma + 2\eta - 1). \tag{17}$$

Therefore, the optimal solution of (16) is clearly bounded by

$$\max\{1 - \frac{\beta}{\sqrt{n+1}+2\beta} \ , \ 1 - \frac{1-\beta}{2(\sqrt{n+1}+\beta)}\} \ \geq \ \mu^{k+1}/\mu^k \ \geq \ 1 - \frac{1-\beta}{2(\sqrt{n+1}+\beta)}. \tag{18}$$

Above analysis points out that for a given $0 < \beta < 1$, using (13), the best reduction rate for μ that the algorithm can achieve is $1 - O(\frac{1}{\sqrt{n}})$. This results in the $O(\sqrt{n}L)$-iteration complexity. It also implies that a better complexity in worst case is very hard to achieve if the 2-norm neighborhood is used. By setting γ and η according to (17), the reduction rate is $1 - \frac{\beta}{\sqrt{n+1}+2\beta}$ when β is near 0, or $1 - \frac{1-\beta}{2(\sqrt{n+1}+\beta)}$ when β is near 1, respectively. Therefore, the algorithm achieves $O(\frac{\sqrt{n}}{\beta(1-\beta)}L)$-iteration complexity.

A simple choice $(y^0 = 0, x^0 = e, \tau^0 = 1, z^0 = e, \kappa^0 = 1)$ ensures that the initial point is $\mathcal{Q}(1,1)$ on the central surface. In summary, we have the following theorem.

Theorem 5 . *Let (LP) have integer data with a total bit length L. Then, (HLF) has integer data with a bit length $O(L)$. Furthermore, let $0 < \beta < 1$ and $(y^0, x^0, \tau^0, z^0, \kappa^0) \in \mathcal{N}(\beta)$ (for instance $(y^0, x^0, \tau^0, z^0, \kappa^0) = (0, e, 1, e, 1)$) and set*

$$\gamma = (\sqrt{n+1}/\beta + 1)(\gamma + 2\eta - 1) > 0$$

$$\alpha^k = \min\{\frac{\sqrt{n+1}+\beta}{(\sqrt{n+1}+2\beta)\gamma} \ , \ \frac{1-\beta}{2\beta\gamma}\} \ > 0.$$

Then, the generalized HSD algorithm generates a strictly complementary optimal solution of (HLF) in $O(\frac{\sqrt{n}}{\beta(1-\beta)}L)$ iterations.

As showed in Goldman and Tucker [3][15], Ye et al. [18], we have the following corollary.

Corollary 6 . *The algorithm specified in Theorem 5 obtains a strictly complementary optimal solution of (LP) and (LD) or detects infeasibility of either (LP) or (LD) in $O(\frac{\sqrt{n}}{\beta(1-\beta)}L)$ iterations.*

4 An HSD algorithm

In this section, we consider a special case of the generalized HSD algorithms. Let $\eta = 0$. The modified Newton equation system becomes

$$\begin{array}{rrrrrcl} & A\ d_x & -b\ d_\tau & & & = & 0, \\ -A^T\ d_y & & +c\ d_\tau & -d_z & & = & 0, \\ b^T\ d_y & -c^T\ d_x & & & -d_\kappa & = & 0, \\ & Z^k\ d_x & & +X^k d_z & & = & \gamma\mu^k e - X^k z^k, \\ & & \kappa^k d_\tau & & +\tau^k d_\kappa & = & \gamma\mu^k - \tau^k\kappa^k. \end{array} \tag{19}$$

We observe that r_P, r_D, r_G disappear in (19). From (12), we have to set $\gamma > 1$. Thus, we can avoid computing these residuals.

Designing algorithms basing on the homogeneous and self-dual linear feasibility model (HLF) seems to have exploited the special properties of linear programming better than on original model. We observe that the introduction of the homogeneous variable τ and the updating of solutions (6) plays an important role in this algorithm. It makes the feasible and infeasible interior-point algorithms with no difference at all. From Lemma 2, clearly, a large γ is desired for this new algorithm, since η is fixed at zero now. In practice, we can make use of a predictor-corrector strategy to choose a very large γ in each iteration, similar to Xu et al. [17] where they used the strategy to choose a very small γ.

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