Contents

Chapter 2

Passband Systems and Analysis

Baseband modulation uses basis functions that have most of their energy at low frequencies. The majority of modulation methods in Chapter 1 are baseband, although a few (like PSK and QAM) use basis functions that have energy centered at or near a carrier or center frequency $\omega_c = 2\pi f_c$. These latter **passband** modulation methods are useful in many applications where transmission occurs over a limited narrow bandwidth, typically centered at or near the carrier frequency of the passband modulation. Digital television transmission on Channel 2 in the US has carrier frequency 52 MHz and non-negligible energy only from 50 to 56 MHz. Digital cellular phones use carrier frequencies in the 900 MHz (and 1800 and 1900 MHz) bands, but have nonzero energy over a narrow band that is typically a few 100 kHz (GSM approximate) to 1.1MHz (wideband "CDMA")) wide.[1] Digital Satellite transmission uses QAM and carriers in the 12 and 17 GHz bands with transponder bandwidths of about 26 MHz. There are numerous other examples. Signal energy is carried through these narrow "passbands" and consequently subject to filtering. This chapter teaches a common method of analyzing such systems without explicit need for the carrier frequency or its inclusion in the basis functions or in the channel transfer function. This theory of **passband system analysis** will allow a fundamental study of the structure of the important suboptimal receivers in Chapter 3 for both baseband and passband modulation.

Passband system analysis is particularly appropriate for quadrature modulation (for instance QAM) signals. It is important to note, and also helps to reinforce the concepts to be introduced, that whether a system is analyzed by using the methods in this chapter or by using the more general methods of Chapter 1, the result should be the same. Section 2.1 introduces the concept of baseband equivalent modeling and shows how to convert from a real signals in a passband channel to complex signals in a baseband equivalent. Section 2.2 then follows with specific consideration of the baseband equivalent filtered AWGN channel that is often used by modem designers, including a detailed discussion of scaling factors often tacitly presumed throughout the industry. Section 2.3 reviews the generation of baseband equivalents and provides an example calculation of the baseband equivalent, which is a reference from which the user may be able to duplicate the procedure in general. Section 2.4 relates complex-signal analysis to the VSB, CAP and OQAM implementations of passband modulation mentioned in Section 1.6.

Especially in passband transmission-system design, the channel is often band-limited so that the AWGN model of Chapter 1 does not directly apply to either the original passband channel or its complex baseband equivalent. Instead, the AWGN model of Chapter 1 is modified to include a filter before the added noise as in Figure 2.1. The filter $h(t)$ represents the band-limiting effect of the physical channel, which may be caused by filters in the transmission path that create the passband channel or by natural finite-bandwidth constraints of transmission lines and wireless multiple-path transmission. The filter $h(t)$ acts on the transmitted modulated signal $x(t)$ and introduces some level of distortion. Preferably, $x(t) * h(t) \approx x(t)$, but the designer may not be able to ensure small distortion. High frequencies are inevitably attenuated in all channels, but many channels also attenuate low frequencies. Furthermore, different frequencies may have different levels of attenuation in real channels. Whether modeling filters

[1] Recent spread spectrum digital wireless has a yet wider bandwidth of about 1.2 MHz wide (CDMA), with carrier frequencies often near 2 Ghz.

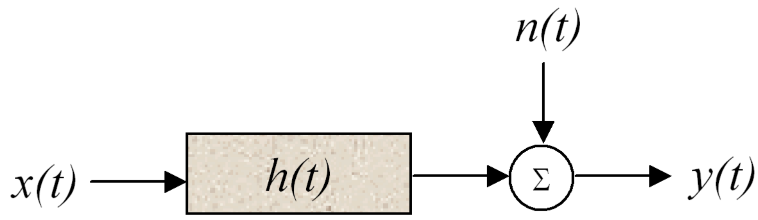

Figure 2.1: The filtered AWGN.

or actual physical effects, an imperfect channel impulse response $h(t)$ will cause an affect the performance of the transmission system.

Chapter 1, Section 7, studied the alterations necessary with such a filtered AWGN for a single "one-shot" transmission. A more general treatment of the widely applying situation modeled by Figure 2.1 for multiple successive message transmission is necessary and appears in Chapters 3, 4 and 5. To prepare for these Chapters, the designer needs understanding of the complex baseband models of this Chapter 2. These models will apply to either the baseband case, where trivially all imaginary components are zero, and to the passband case where all imaginary components are not necessarily zero, allowing a single theory of receiver processing of complex signals in the remainder of this text.

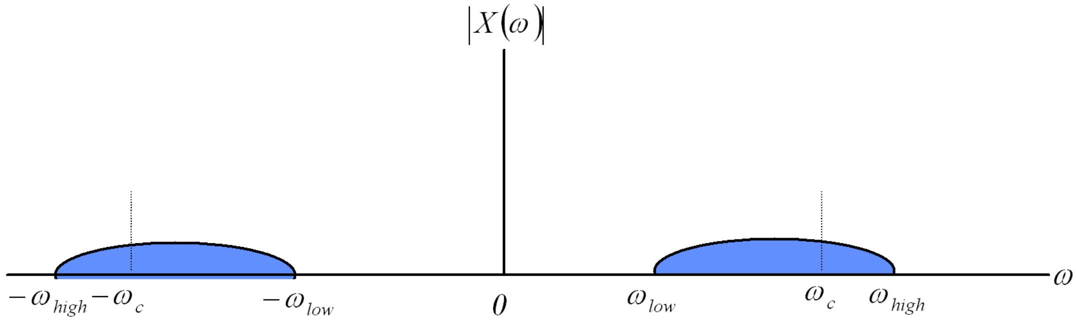

Figure 2.2: Passband signal amplitude.

2.1 Passband Representations and Terminology

A passband signal has energy concentrated in the vicinity of a frequency $\omega_c = 2\pi f_c$ in anticipation of transmission through a passband channel that only passes energy in this same frequency band. Thus, passband signals are designed for passband channels.

Passband signals usually have been generated through multiplication of a "lowpass" signal by a sinusoid to move the energy away from low frequencies towards the frequency band around ω_c. Such passband modulation is used on channels that do not pass DC or on channels that several signals simultaneously share in non-overlapping frequency bands.

This section first investigates a number of equivalent representations of a passband signal, the most interesting of which is the baseband equivalent signal in Subsection 2.1.1. The designer replaces the original modulated passband signal with the baseband-equivalent signal in most modern transmission analyzes. The objective in Subsection 2.1.1 is to teach the reader to generate such an equivalent signal from the original signal. Subsection 2.1.2 studies the frequency content of baseband-equivalent signals, essentially showing that amplitude is doubled and translated to DC. Since all baseband signals are centered at DC, a common baseband processing method can be applied, for any passband channel, as in Subsection 2.1.3. Figure 2.7 is a quick summary of this entire section.

2.1.1 Passband Signal Equivalents

The real-valued signal $x(t)$ is a **passband signal** when its nonzero fourier transform is near ω_c, as in Figure 2.2. Passband signals never have DC content, so $X(0) = 0$.

Definition 2.1.1 (Carrier-Modulated Signal) *A* **carrier-modulated signal** *is any passband signal that can be written in the following form*

$$x(t) = a(t)\cos\left(\omega_c t + \theta(t)\right) \quad , \tag{2.1}$$

where $a(t)$ is the time-varying **amplitude** *or* **envelope** *of the modulated signal and $\theta(t)$ is the time-varying* **phase**. *ω_c is called the* **carrier frequency** *(in radians/sec).*

The carrier frequency ω_c is chosen sufficiently large compared with the amplitude and phase variations of $a(t)$ so that the power spectral density does not have significant energy at $\omega = 0$. See Figure 2.2, wherein the spectrum of $X(\omega)$ is concentrated in the passband $\omega_{low} < |\omega| < \omega_{high}$.

In digital communication, $x(t)$ is equivalently written in quadrature form using the trigonometric identity $\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v)$, leading to a quadrature decomposition:

Definition 2.1.2 (Quadrature Decomposition) *The* **quadrature decomposition** *of a carrier modulated signal is*

$$x(t) = x_I(t)\cos\left(\omega_c t\right) - x_Q(t)\sin\left(\omega_c t\right) \quad , \tag{2.2}$$

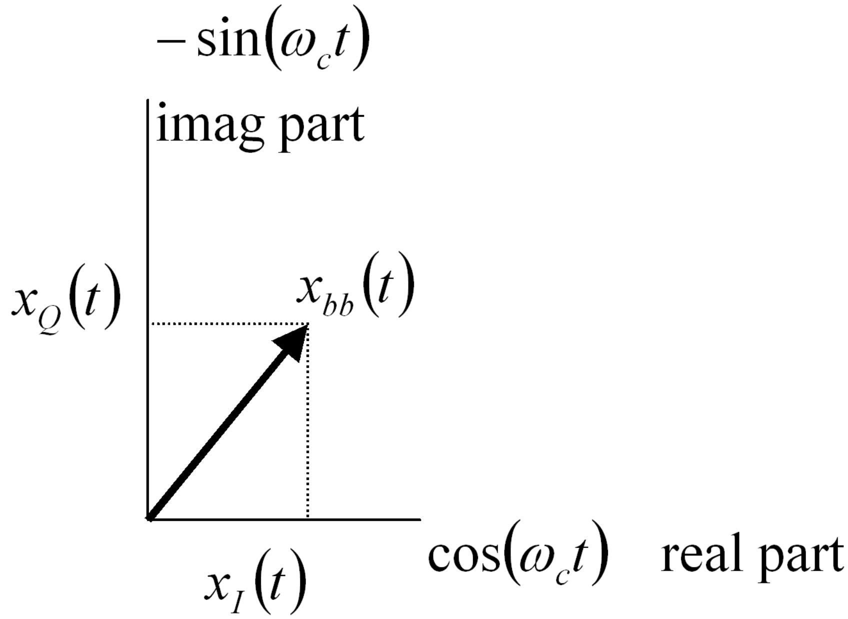

Figure 2.3: Decomposition of baseband-equivalent signal.

where $x_I(t) = a(t)\cos(\theta(t))$ *is the time-varying* **inphase component** *of the modulated signal, and* $x_Q(t) = a(t)\sin(\theta(t))$ *is the time-varying* **quadrature component**.

Relationships determining $(a(t),\ \theta(t))$ from $(x_I(t),\ x_Q(t))$ are

$$a(t) = \sqrt{x_I^2(t) + x_Q^2(t)} \quad , \tag{2.3}$$

and

$$\theta(t) = \mathrm{Tan}^{-1}\left[\frac{x_Q(t)}{x_I(t)}\right] \quad . \tag{2.4}$$

In (2.4), the inverse tangent is taken with the polarities of the numerator and denominator independently known, so there is no quadrant ambiguity in computing $\theta(t)$.

In passband processing and analysis, the objective is to eliminate explicit consideration of the carrier frequency ω_c and directly analyze systems using only the inphase and quadrature components. These inphase and quadrature components can be combined into a two-dimensional vector, or into an equivalent complex signal. By convention, a graph of a quadrature-modulated signal plots the inphase component along the real axis and the quadrature component along the imaginary axis as shown in Figure 2.3. The resultant complex vector $x_{bb}(t)$ is known as the complex baseband-equivalent signal.

Definition 2.1.3 (Baseband-Equivalent Signal) *The complex* **baseband-equivalent signal** *for* $x(t)$ *in (2.1) is*

$$x_{bb}(t) \stackrel{\Delta}{=} x_I(t) + \jmath x_Q(t) \quad , \tag{2.5}$$

where $\jmath = \sqrt{-1}$.

The baseband-equivalent signal expression no longer explicitly contains the carrier frequency ω_c. Another complex representation that does explicitly contain ω_c is the analytic equivalent signal for $x(t)$:

Definition 2.1.4 (Analytic-Equivalent Signal) *The* **analytic-equivalent signal** *for* $x(t)$ *in (2.1) is*

$$x_A(t) \stackrel{\Delta}{=} x_{bb}(t)e^{\jmath\omega_c t} \quad . \tag{2.6}$$

The original real-valued passband signal $x(t)$ is the real part of the analytic equivalent signal:

$$x(t) = \Re\{x_A(t)\} \quad . \tag{2.7}$$

The Hilbert transform of $x(t)$, denoted by $\check{x}(t)$, is the imaginary part of the analytic signal as

$$\check{x}(t) = \Im\{x_A(t)\} \quad . \tag{2.8}$$

(See Appendix A for more details on the Hilbert transform and a proof of (2.8).) Finally, the inphase component $x_I(t)$ and the quadrature component $x_Q(t)$ can be expressed using the signal $x(t)$ and its Hilbert transform $\check{x}(t)$ as (using $x_{bb}(t) = x_I(t) + \jmath x_Q(t) = x_A(t) \cdot e^{-\jmath\omega_c t}$):

$$\begin{aligned} x_I(t) &= x(t)\cos(\omega_c t) + \check{x}(t)\sin(\omega_c t) && (2.9) \\ x_Q(t) &= \check{x}(t)\cos(\omega_c t) - x(t)\sin(\omega_c t) \quad . && (2.10) \end{aligned}$$

Thus, four equivalent forms for representing a real passband signal $x(t)$ with carrier frequency ω_c are:

$$\begin{array}{lc} \text{1. magnitude, phase} & a(t), \theta(t) \\ \text{2. inphase, quadrature} & x_I(t), x_Q(t) \\ \text{3. complex baseband} & x_{bb}(t) \\ \text{4. analytic} & x_A(t) \end{array} \tag{2.11}$$

EXAMPLE 2.1.1 (Translation between equivalent representations:) A passband QAM signal is

$$x(t) = \text{sinc}(10^6 t) \cdot \cos(2\pi 10^7 t) + 3\text{sinc}(10^6 t) \cdot \sin(2\pi 10^7 t) \quad . \tag{2.12}$$

The carrier frequency is 10 MHz and the symbol period is 1 μs. The inphase and quadrature components are

$$\begin{aligned} x_I(t) &= \text{sinc}(10^6 t) && (2.13) \\ x_Q(t) &= -3 \cdot \text{sinc}(10^6 t) \quad , && (2.14) \end{aligned}$$

so

$$x_{bb}(t) = (1 - 3\jmath) \cdot \text{sinc}(10^6 t) \quad . \tag{2.15}$$

The amplitude and phase of the complex baseband signal are

$$\begin{aligned} a(t) &= \sqrt{10} \cdot \text{sinc}(10^6 t) && (2.16) \\ \theta(t) &= \text{Tan}^{-1}\left[\frac{-3}{1}\right] = -71.6^o \quad . && (2.17) \end{aligned}$$

Thus,

$$x(t) = \sqrt{10} \cdot \text{sinc}(10^6 t) \cdot \cos(\omega_c t - 71.6^o) \quad . \tag{2.18}$$

Finally,

$$x_A(t) = (1 - 3\jmath) \cdot \text{sinc}(10^6 t) \cdot e^{\jmath 2\pi 10^7 t} \quad . \tag{2.19}$$

Section 2.1.2 next considers the relationship of the Fourier transforms of $x(t), x_{bb}(t),$ and $x_A(t)$.

2.1.2 Frequency Spectrum of Analytic- and Baseband-Equivalent Signals

Using Equations (2.7) and (2.8) the analytic signal is represented as shown in Figure 2.4.

$$x_A(t) = x(t) + \jmath\check{x}(t) \quad . \tag{2.20}$$

Taking the Fourier Transform of both sides of (2.20) yields[2]

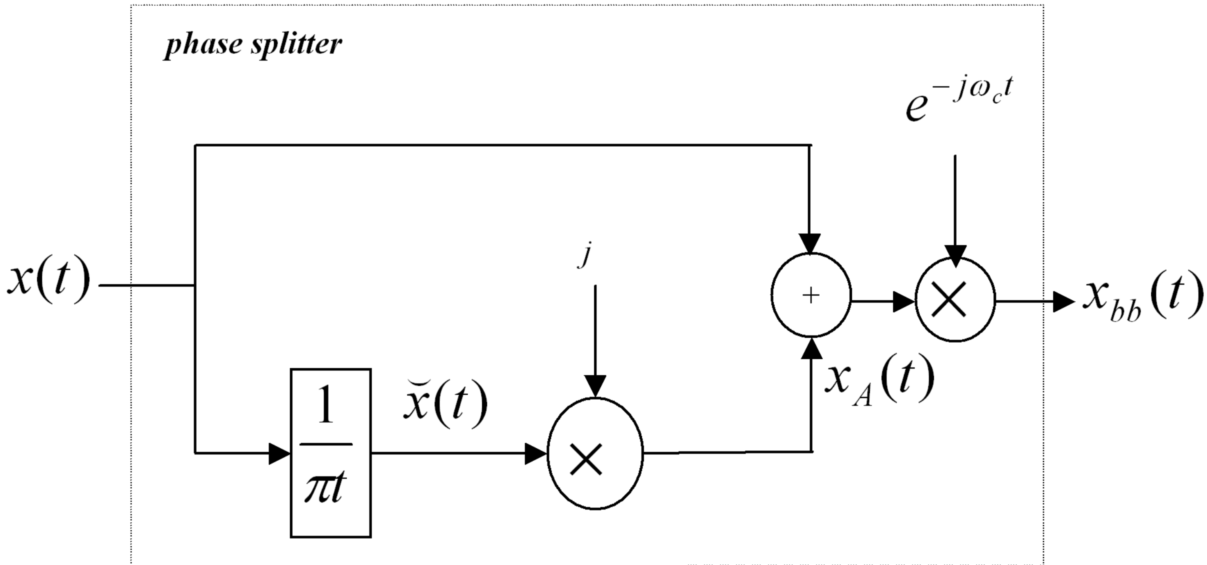

Figure 2.4: Analytic signal composition.

$$
\begin{aligned}
X_A(\omega) &= [1+\mathrm{sgn}(\omega)]\,X(\omega) && (2.21)\\
&= \begin{cases} 2X(\omega) & \omega > 0 \\ X(0) = 0 & \omega = 0 \\ 0 & \omega < 0 \end{cases} \quad . && (2.22)
\end{aligned}
$$

The analytic equivalent signal, $x_A(t)$, contains only the positive frequencies of $x(t)$ and is identically zero for negative frequencies. The Fourier transform $X(\omega)$ of the real signal $x(t)$ has two symmetry properties: The real part $\mathcal{R}\{X(\omega)\}$ is even in ω, while the imaginary part $\mathcal{I}\{X(\omega)\}$ is odd in ω. Knowledge of only the non-negative frequencies of $X(\omega)$, such as are supplied by the analytic signal, is sufficient for reconstruction of $X(\omega)$. Thus, one confirms that the analytic signal $x_A(t)$ is truly "equivalent" to the original signal $x(t)$.

Using Equation (2.6), the Fourier transform of the baseband equivalent signal is simply the Fourier transform of the analytic signal translated in frequency ω_c. Thus

$$X_A(\omega) = X_{bb}(\omega - \omega_c) \tag{2.23}$$

and

$$X_{bb}(\omega) = X_A(\omega + \omega_c) \quad . \tag{2.24}$$

Use of (2.6) and (2.7) allows reconstruction of the signal $x(t)$ from the baseband equivalent signal $x_{bb}(t)$ and the carrier frequency ω_c. The baseband equivalent signal, in general, may be complex-valued, and thus as shown in Figure 2.5 the spectrum of $x_{bb}(t)$ may be asymmetric about the origin $\omega = 0$.

EXAMPLE 2.1.2 (Continuing Example) Figure 2.6 shows the original, baseband, and analytic equivalent spectra of the signal

$$x(t) = \mathrm{sinc}(10^6 t)\cdot\cos(2\pi 10^7 t) + 3\mathrm{sinc}(10^6 t)\cdot\sin(2\pi 10^7 t) \quad . \tag{2.25}$$

The doubling in amplitude of the two complex signals' Fourier transforms occurs because all energy in these complex representations appears in a single positive-frequency band.

[2] If $\breve{x}(t)$ is the Hilbert transform of $x(t)$, then the Fourier transform of $\breve{x}(t)$ is $-\jmath\mathrm{sgn}(\omega)X(\omega)$, where $X(\omega)$ is the Fourier Transform of $x(t)$, as shown in Appendix A.

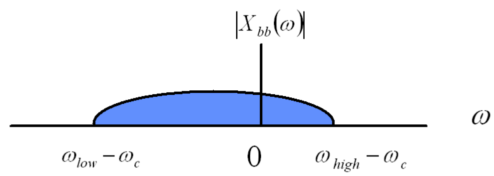

Figure 2.5: Baseband signal spectrum.

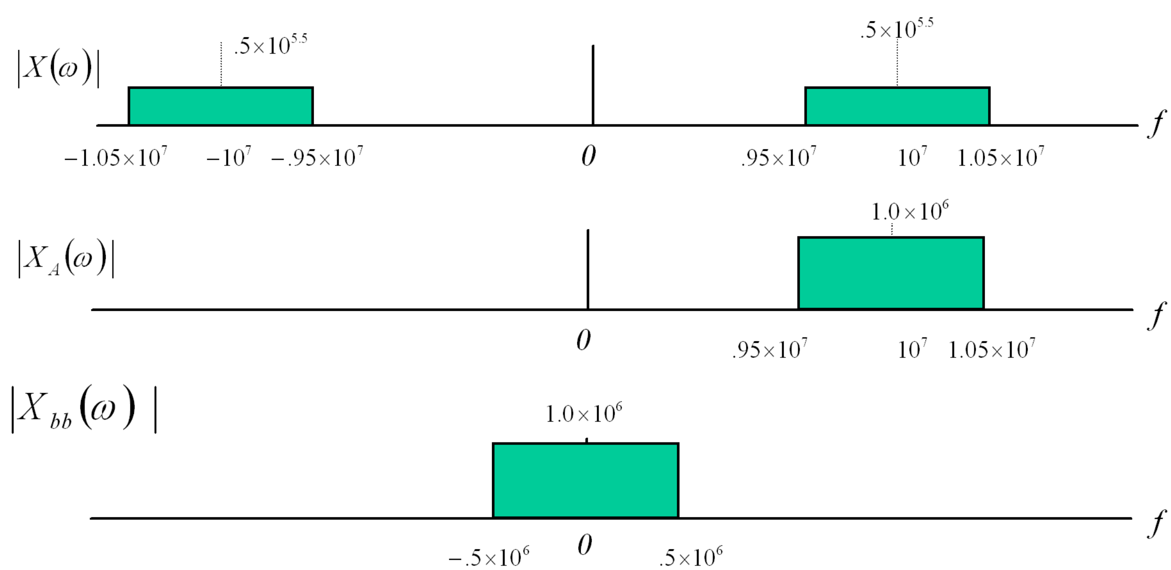

Figure 2.6: Spectra equivalents for example waveform.

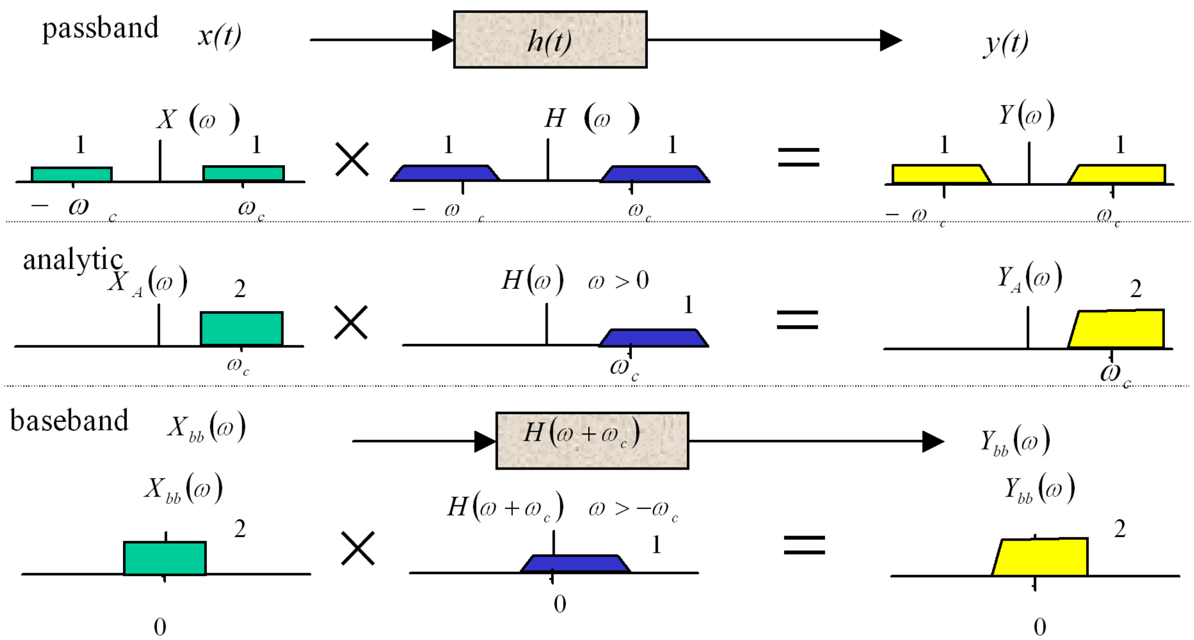

Figure 2.7: Equivalent representations of the channel input/output relationship.

Generation of the baseband equivalent

To generate the baseband equivalent of a signal, the structure in Figure 2.4 is used, where the second complex multiply simply is 4 real multiplies using Euler's formula $e^{\jmath\omega_c t} = \cos(\omega_c t) + \jmath\sin(\omega_c t)$. The first multiply by $\jmath$ alone is, of course, symbolic and simply means that the receiver processing views the signal on that path as the imaginary part in complex arithmetic.

2.1.3 Passband Channels

The designer would like a relationship between the baseband equivalent modulated signal on the input and output of a filtered passband channel. Such a relationship is easily computed and essentially is the multiplication of the baseband input spectra by the transform of the channel translated from ω_c to DC, as this section shows.

The complex baseband representation can be used to characterize the input/output relationships of passband signals and channels. In particular, if the passband signal $x(t)$ is filtered by a passband linear channel with impulse response $h(t)$, passband analysis directly determines the baseband- and analytic-equivalent representations of the filter-output signal $y(t)$ as well as of the channel $h(t)$. Figure 2.7 summarizes the final results that this section shall derive.

Equivalent representations of the channel response.

Any of the four representations found for passband signals in the previous section apply to the impulse response or Fourier transform for the channel by using the same equations and substituting h for x. For instance, linear time-invariant channels can be described by a real-valued impulse response $h(t)$. For any $h(t)$, the **analytic-equivalent channel** $h_A(t)$ is

$$h_A(t) \stackrel{\Delta}{=} h(t) + \jmath\check{h}(t) \quad . \tag{2.26}$$

Similarly in the frequency domain,

$$H_A(\omega) = [1 + \mathrm{sgn}(\omega)]\, H(\omega) \quad . \tag{2.27}$$

The **baseband-equivalent channel** is defined in the same manner as a baseband equivalent signal, except the carrier frequency ω_c is set equal to that of the input, and output, signals.

Definition 2.1.5 (Baseband Equivalent Channel (at carrier frequency ω_c)) *The* **baseband equivalent channel** *at any carrier frequency ω_c is given by*

$$h_{bb}(t) \stackrel{\Delta}{=} h_A(t) \cdot e^{-\jmath\omega_c t} \quad . \tag{2.28}$$

For valid application of the term "baseband equivalent", the carrier frequency should be sufficiently large to guarantee that $h_{bb}(t)$ has no significant energy content at frequencies $|\omega| > \omega_c$,i.e., $|H_{bb}(\omega)| = 0 \ \forall \ |\omega| > \omega_c$.

The equivalent views of channel input/output relations

The frequency-domain representation of the passband system at the top of Figure 2.7 is,

$$Y(\omega) = H(\omega) \cdot X(\omega) \quad . \tag{2.29}$$

Multiplying both sides of (2.29) by $1 + sgn(\omega)$ leads to (middle of Figure 2.7)

$$\begin{aligned} Y_A(\omega) &= H(\omega) \cdot X_A(\omega) && (2.30)\\ \Rightarrow Y_A(\omega) &= [H(\omega) \cdot \frac{1}{2} \cdot (1 + sgn(\omega))] \cdot X_A(\omega) = \left[\frac{1}{2} \cdot H_A(\omega)\right] \cdot X_A(\omega), && (2.31) \end{aligned}$$

where the second relationship follows by observing that since the input has nonzero spectra only for positive frequencies, one may then look at the action of the channel only at those same positive frequencies (recalling that the factor $(1/2)\cdot[1+sgn(\omega)]$ extracts the positive frequencies). More importantly, since the linear time-invariant passband channel $h(t)$ only scales and phase shifts each frequency independently, the output $y(t)$ has its power spectral density concentrated in the same frequency region (or a smaller region if the channel zeroes a band) as the input $x(t)$. Shift of the output spectrum $y(t)$ down by ω_c yields

$$\begin{aligned} Y_{bb}(\omega) &= Y_A(\omega + \omega_c) = \left[\frac{1}{2} H_A(\omega + \omega_c)\right] X_A(\omega + \omega_c) && (2.32)\\ &= \left[\frac{1}{2} H_{bb}(\omega)\right] X_{bb}(\omega) && (2.33)\\ &= H(\omega + \omega_c) \cdot X_{bb}(\omega) \ \ \omega > -\omega_c \ \ , && (2.34) \end{aligned}$$

which is also illustrated at the bottom of Figure 2.7. This leads to the definition of the **baseband equivalent system**

Definition 2.1.6 (Baseband Equivalent System) *The* **baseband equivalent system** *for a passband system described by $y(t) = x(t) * h(t)$, where $x(t)$ is a passband signal, is given by*

$$y_{bb}(t) = \left(x_{bb}(t) * \frac{1}{2} h_{bb}(t)\right) \tag{2.35}$$

or

$$Y_{bb}(\omega) = H(\omega + \omega_c) \cdot X_{bb}(\omega) \quad . \tag{2.36}$$

Obtaining the baseband equivalent channel is easy! Simply slide the Fourier transform of the channel response down to DC. Because the channel may be asymmetric with respect to ω_c, the baseband equivalent channel can be complex and usually is. The complexity of dealing with cosines, sines, and carrier frequencies is removed by the baseband-equivalent representation. Any channel with any carrier frequency can thus be represented in a common baseband framework, which will be convenient for many analyzes in digital transmission. This is why baseband-equivalent channels dominate in their use in digital transmission analysis. The baseband-equivalent input is convolved with the complex channel corresponding to $H(\omega + \omega_c)$ to get the baseband-equivalent output.

A channel that is not passband, but rather initially real baseband, simply corresponds to the baseband equivalent input/output representation with all imaginary parts zeroed, and $H(\omega)$ used directly ($\omega_c = 0$).

The input/output relationships can thus be summarized as follows: For the passband signals and systems,

$$\begin{aligned} y(t) &= x(t) * h(t) && (2.37) \\ Y(\omega) &= X(\omega) \cdot H(\omega). && (2.38) \end{aligned}$$

For the analytic-equivalent system,

$$\begin{aligned} y_A(t) &= x_A(t) * \frac{1}{2} h_A(t) && (2.39) \\ Y_A(\omega) &= X_A(\omega) \cdot H(\omega). && (2.40) \end{aligned}$$

For the baseband equivalent system,

$$\begin{aligned} y_{bb}(t) &= x_{bb}(t) * \frac{1}{2} h_{bb}(t) && (2.41) \\ Y_{bb}(\omega) &= X_{bb}(\omega) \cdot H(\omega + \omega_c). && (2.42) \end{aligned}$$

Any of these three equivalent relations (and ω_c) fully describe the passband system.

EXAMPLE 2.1.3 (Bandpass channel for previous bandpass signals) A channel impulse response is $h(t) = 2 \times 10^6 \text{sinc}(10^6 t) \cdot \cos(2\pi 10^7 t)$ corresponds to

$$H(f) = \begin{cases} 1 & |f \pm 10^7| < .5 \times 10^6 \\ 0 & \text{elsewhere} \end{cases} \quad . \tag{2.43}$$

Then

$$h_I(t) = 2 \times 10^6 \cdot \text{sinc}(10^6 t)$$

and

$$h_Q(t) = 0 \quad ,$$

so that $h_{bb}(t) = h_I(t)$ or

$$H_{bb}(f) = \begin{cases} 2 & |f| \le 500 \text{ kHz} \\ 0 & \text{elsewhere} \end{cases} \quad .$$

Using the signal in Examples 2.1.1 and 2.1.2 as the channel input, the channel output is

$$Y_{bb}(f) = H(f + f_c) \cdot X_{bb}(f) = 1 \cdot X_{bb}(f) \quad |f| < 500 \text{ kHz} \tag{2.44}$$

or

$$x_{bb}(t) * \frac{1}{2} h_{bb}(t) = x_{bb}(t) = (1 - 3\jmath) \cdot \text{sinc}(10^6 t) \quad . \tag{2.45}$$

Zero quadrature channel or $h_Q(t) = 0$ does not mean that no quadrature signal components are passed – it means that inphase components remain inphase components at the channel output, and quadrature components similarly then remain quadrature components at the channel output.

Appendix B extends the results of this section to passband random processes (this section only considered deterministic signals) used on passband deterministic channels.

The next section considers the addition of random noise to the channel output.

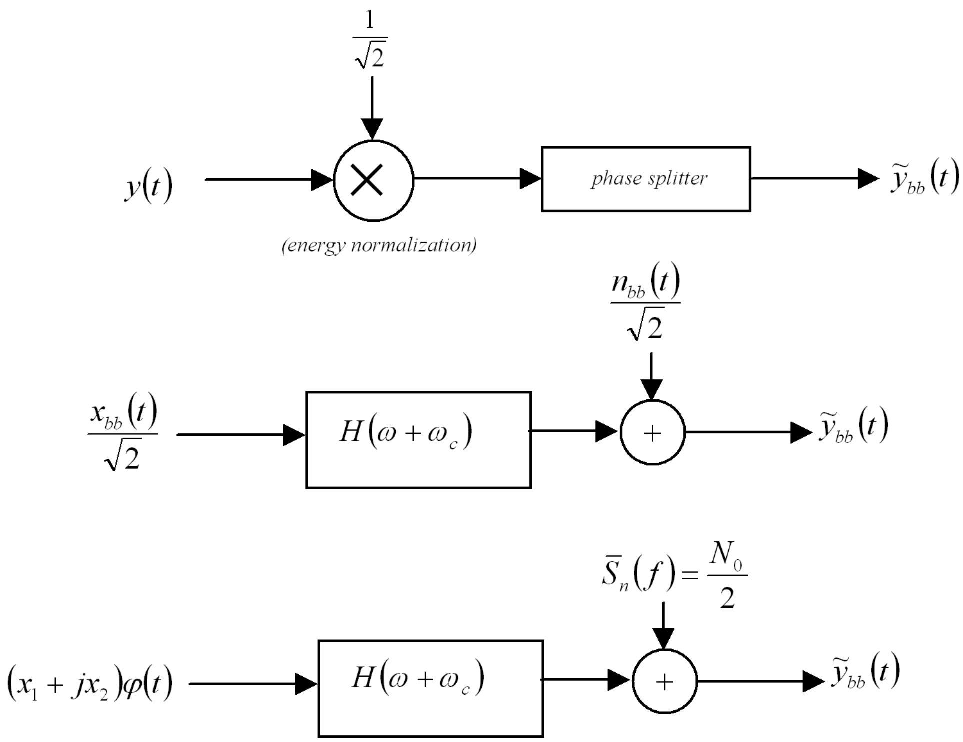

Figure 2.8: Scaling for passband white Gaussian noise

2.2 Baseband-Equivalent AWGN Channel

This section investigates the processing of a passband filtered AWGN channel. Some results from Appendix B on passband random processes appear and the reader may best read that appendix first before proceeding, although such reading is not completely necessary. Figure 2.8 summarizes a scaling factor used, explicity or tacitly, by all developments based on passband processes. This scale factor is simply for analytical purposes and will make results consistent in all regards with those in Chapter 1.

The channel output $y(t)$ is processed by the combination of a scale factor $1/\sqrt{2}$ and a phase splitter to generate a baseband equivalent signal $\tilde{y}_{bb}(t)$. The tilde is often dropped in the literature and the output of the splitter is often called $y_{bb}(t)$ even though the scale factor is included. Generally speaking, the phase splitter adds a signal to j times its own Hilbert transform. The Hilbert transform of a random process has the same power as the original process (Appendix B). Thus, the phase splitter generally doubles power somewhat arbitrarily. The scale factor of $1/\sqrt{2}$ in front of the phase splitter causes the power to appear the same. The power-scaling occurs equally for both the noise and signal, since both are present in the channel output $y(t)$, and so performance is not changed no matter what the scale factor and the ratio of minimum distance to noise standard deviation is also unchanged by any scaling. Nonetheless, the particular scaling is often presumed in phase splitters simply because the ensuing analysis is consistent with the types of scalings also tacitly used in Chapter 1.

For analysis, the scale factor can be "pushed back" through the channel, and the baseband-equivalent system in the middle of Figure 2.8 is equivalent to the upper figure with the explicit scaling. The scale factor then occurs separately in each of the noise and signal components of the output. Each of the next two subsections independently investigates this scaling and justifies its consistency with previous results.

2.2.1 Noise scaling in the baseband AWGN

Appendix B shows that the power spectral density of the analytic equivalent of a random process is equal to four times the power spectral density of the positive-frequency part of the original processing, which tacitly implies a doubling of noise power.[3]

Since the scaled WGN, $n(t)/\sqrt{2}$, in Figure 2.8 has power spectral density.

$$\mathcal{S}_n(\omega) = \frac{\mathcal{N}_0}{4} \quad , \tag{2.46}$$

the power spectrum of the analytic equivalent of $n(t)/\sqrt{2}$ is

$$S_A(\omega) = \begin{cases} \mathcal{N}_0 & \omega > 0 \\ \frac{\mathcal{N}_0}{2} & \omega = 0 \\ 0 & \omega < 0 \end{cases} \quad . \tag{2.47}$$

The baseband equivalent noise has power spectrum that simply translates $S_A(\omega)$ to baseband, or

$$S_{bb}(\omega) = \begin{cases} \mathcal{N}_0 & \omega > -\omega_c \\ \frac{\mathcal{N}_0}{2} & \omega = -\omega_c \\ 0 & \omega < -\omega_c \end{cases} \quad . \tag{2.48}$$

Strictly speaking, $S_A(\omega)$ and $S_{bb}(\omega)$ do not correspond to white noise. However, practical systems will always use a carrier frequency that is at least equal to the signal frequency, ω_{high}, that corresponds to the highest-frequency nonzero baseband signal component – that is, the design always modulates with a carrier frequency large enough to "get away" from DC. In this case, the power spectrum of the baseband equivalent noise appears as if it were "white" or flat at $\mathcal{N}_0$ for all frequencies of practical interest. The noise in this baseband demodulated signal is complex AWGN with power spectral density $\mathcal{N}_0$. This is generally accepted practice in digital communication.

Whenever the scaled phase-splitting arrangement of Figure 2.8 is used, this text defines baseband equivalent WGN as follows:

> **Definition 2.2.1 (Baseband Equivalent WGN) Baseband Equivalent White Gaussian Noise** *is a random process, $\tilde{n}_{bb}(t)$, that is generated, essentially, through demodulation of the Passband AWGN in Figure 2.8. The autocorrelation of the complex random process, $r_{bb}(\tau)$ is thus defined to be*
>
> $$r_{bb}(\tau) = \mathcal{N}_0 \cdot \delta(\tau) \quad , \tag{2.49}$$
>
> *and the power spectral density is thus*
>
> $$S_{bb}(f) = \mathcal{N}_0 \quad . \tag{2.50}$$

From Appendix B, the baseband autocorrelation is then

$$r_{bb}(\tau) = 2r_I(\tau) = 2r_Q(\tau) = \mathcal{N}_0\delta(\tau) \quad , \tag{2.51}$$

so that the inphase and quadrature noises each have power spectral density $\frac{\mathcal{N}_0}{2}$ and are white noise signals. Further, from Appendix B and (2.51),

$$r_{IQ}(\tau) = 0 \quad , \tag{2.52}$$

that is, the inphase and quadrature noises are uncorrelated for all time lags with baseband equivalent WGN.

The complex baseband noise is two dimensional (two real dimensions), and the noise variance per dimension is thus $\frac{\mathcal{N}_0}{2}$, which is the reason for the scaling that was introduced in the definition of passband WGN. This scaling makes the noise variance per dimension the same as discussed in Chapter 1.

[3] The autocorrelation of the analytic equivalent noise is $r_A(\tau) = 2\left(r_n(\tau) + \jmath\check{r}_n(\tau)\right)$. See Appendix B for more details.

2.2.2 Scaling of the Signal

A brief review of the basis-function modulator of Chapter 1 will assist understanding of the effect of the scaling: The two normalized QAM passband functions for transmission on the one-shot AWGN are again

$$\varphi_1(t) = \sqrt{2}\varphi(t)\cos(\omega_c t) \tag{2.53}$$

$$\varphi_2(t) = -\sqrt{2}\varphi(t)\sin(\omega_c t) \quad , \tag{2.54}$$

where for practical reasons, ω_c is high enough.[4] The modulated signal

$$x(t) = x_1\varphi_1(t) + x_2\varphi_2(t) \tag{2.55}$$

$$= \sqrt{2}\,\{x_1\varphi(t)\cos(\omega_c t)\} - \sqrt{2}\,\{x_2\varphi(t)\sin(\omega_c t)\} \quad , \tag{2.56}$$

has baseband equivalent signal

$$x_{bb}(t) = \sqrt{2}\,(x_1 + \jmath x_2)\cdot\varphi(t) \quad . \tag{2.57}$$

The scaling of Figure 2.8 removes the extra factor of $\sqrt{2}$ that arose through normalization of the modulated basis function. The bottom diagram in Figure 2.8 shows this removal explictly so that the system appears as a complex baseband system with complex input

$$\tilde{x}_{bb}(t) = (x_1 + \jmath x_2)\cdot\varphi(t) \quad . \tag{2.58}$$

Equation (2.58) becomes

$$\tilde{x}_{bb}(t) = \boldsymbol{x}_{bb}\cdot\varphi(t) \tag{2.59}$$

where

$$\boldsymbol{x}_{bb} \stackrel{\Delta}{=} (x_1 + \jmath x_2) \quad . \tag{2.60}$$

Equations (2.59) and (2.60) constitute a single-dimension **complex** baseband representation of the QAM modulator with (now normalized) basis function $\varphi(t)$ that is entirely consistent in all regards with the two-real-dimensional representation of Chapter 1. The average energy of the complex signal constellation is

$$\mathcal{E}_{bb} = \mathcal{E}_{\boldsymbol{x}} = 2\bar{\mathcal{E}}_{\boldsymbol{x}} \quad , \tag{2.61}$$

which maintains the convention that a complex signal is equivalent to a two-dimensional real signal in defining $\bar{\mathcal{E}}_{\boldsymbol{x}}$.

The tildes are necessary for those learning baseband analysis, but are dropped without comment throughout the literature. So one often sees a complex AWGN defined by Figure 2.9, where the tildes are dropped, but the scaling of noise and signal are consistent with the $1/\sqrt{2}$ in Figure 2.8.

Furthermore, this figure is often used to represent one-dimensional real systems where no passband modulation effects are of concern. In this case, the quadrature (imaginary) dimension is tacitly zeroed, while the real dimension carries the signal and has noise power spectral density $\frac{\mathcal{N}_0}{2}$, entirely consistent with Chapter 1.

[4] One verifies that these functions are indeed normalized – if $\varphi(t)$ is normalized, as we assume – by investigating their power spectra under modulation.

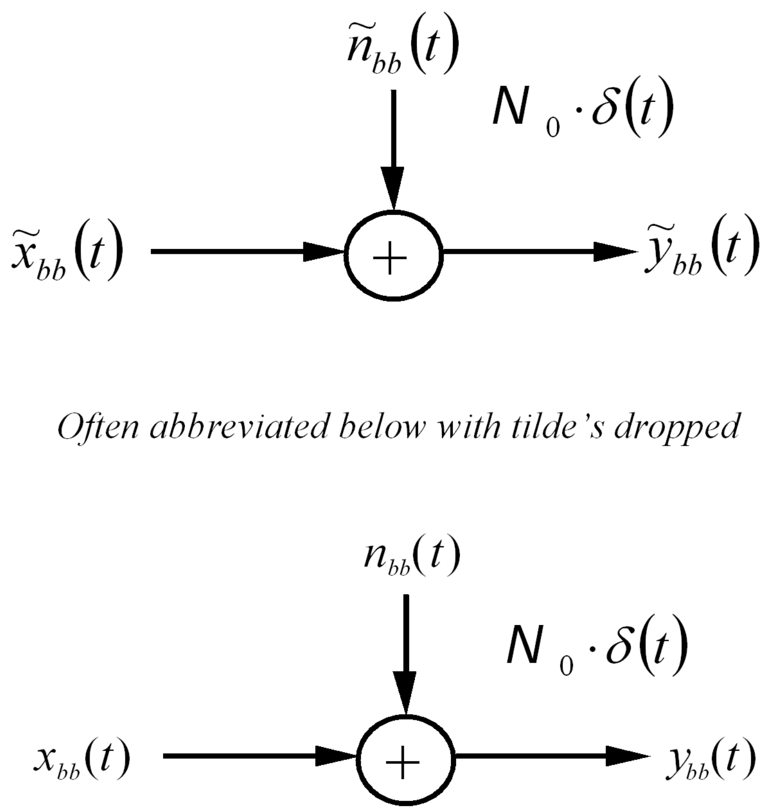

Figure 2.9: Baseband AWGN

2.3 Conversion to a baseband equivalent channel

Baseband analysis is applied to a QAM system where the channel and noise have been modeled as equivalent baseband. The system then looks like a PAM system except that inputs, outputs and internal quantities are all complex with the real dimension corresponding to the "cosine" modulated component and the imaginary dimension corresponding to the "sine" modulated component. After moving to a complex baseband equivalent, the effect of the carrier has been removed from all subsequent analysis.

2.3.1 Demodulators for the generation of the baseband equivalent

The actual generation of the baseband equivalent output signal can take one of the two equivalent forms in Figure 2.10. Figure 2.10(a) repeats the "phase-splitter" generation of the analytic equivalent, $y_A(t)$. The scale factor is of course absorbed into the definition of noise power spectral density and baseband input modulator for convenient analysis, as in the last section. The analytic signal $y_A(t)$ is demodulated by $e^{-j\omega_c t}$ to generate $y_{bb}(t)$. Figure 2.10(b) illustrates a more obvious form of generating $y_{bb}(t)$ that is sometimes used in practice. The structure in Figure 2.10(b) generates the inphase and quadrature components, $y_I(t)$ and $y_Q(t)$ by multiplying $y(t)$ by $2\cos(\omega_c(t))$ and $2\sin(\omega_c(t))$ in parallel. Then,

$$\begin{aligned}2\cos(\omega_c(t))y(t) &= y_I(t)2\cos(\omega_c(t))^2 - y_Q(t)2\sin(\omega_c(t))\cos(\omega_c(t)) \\ &= y_I(t)(1+\cos(2\omega_c t)) - y_Q(t)\sin(2\omega_c(t))\end{aligned}$$

and

$$\begin{aligned}2\sin(\omega_c(t))y(t) &= y_I(t)2\cos(\omega_c(t))\sin(\omega_c(t)) - y_Q(t)2\sin(\omega_c(t))^2 \\ &= y_I(t)\sin(2\omega_c t) + y_Q(t)(\cos(2\omega_c(t)) - 1)\end{aligned}$$

Lowpass filtering of $2\cos(\omega_c(t))y(t)$ and $2\sin(\omega_c(t))y(t)$ removes the signal artifacts centered at $2\omega_c$. In practice, it is usually easier to implement the two identical lowpass filters in Figure 2.10 (b) than the Hilbert filter in Figure 2.10 (a); so the implementation shown in Figure 2.10 (b) may be preferred in simple communication systems. In more sophisticated designs, especially those involving equalization (see Chapter 3), the implementation in Figure 2.10 (a) can have some practical performance advantages for the implementation of carrier-phase recovery systems (see Chapter 6).

EXAMPLE 2.3.1 (demodulation of a specific signal) As in Example 2.1.1 from Section 2.1, a passband AWGN-channel output QAM signal is

$$z(t) = \mathrm{sinc}(10^6 t)\cdot\cos(2\pi 10^7 t) + 3\cdot\mathrm{sinc}(10^6 t)\cdot\sin(2\pi 10^7 t) + n(t) \quad . \tag{2.62}$$

The carrier frequency is 10 MHz and the symbol period is 1 μs. Proceeding through the demodulator in Figure 2.10(a), $z(t)$ is the channel output signal that is input to the demodulator. The signal after scaling by $\frac{1}{\sqrt{2}}$ is

$$y(t) = \frac{z(t)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\cdot\mathrm{sinc}(10^6 t)\cdot\cos(2\pi 10^7 t) + \frac{3}{\sqrt{2}}\cdot\mathrm{sinc}(10^6 t)\cdot\sin(2\pi 10^7 t) + \frac{n(t)}{\sqrt{2}} \quad , \tag{2.63}$$

so that effectively this choice of location for defining $y(t)$ allows the $1/\sqrt{2}$ scaling factor to be viewed as absorbed into the channel. The Hilbert transform of this scaled signal for the lower path in parallel in Figure 2.10(a) is

$$\check{y}(t) = \frac{1}{\sqrt{2}}\cdot\mathrm{sinc}(10^6 t)\cdot\sin(2\pi 10^7 t) - \frac{3}{\sqrt{2}}\cdot\mathrm{sinc}(10^6 t)\cdot\cos(2\pi 10^7 t) + \check{n}(t) \quad . \tag{2.64}$$

Multiplying the Hilbert transform by $\jmath$ and adding to the upper unchanged component ($y(t)$) creates the analytic signal

$$y_A(t) = y(t) + \jmath\check{y}(t) \quad , \tag{2.65}$$

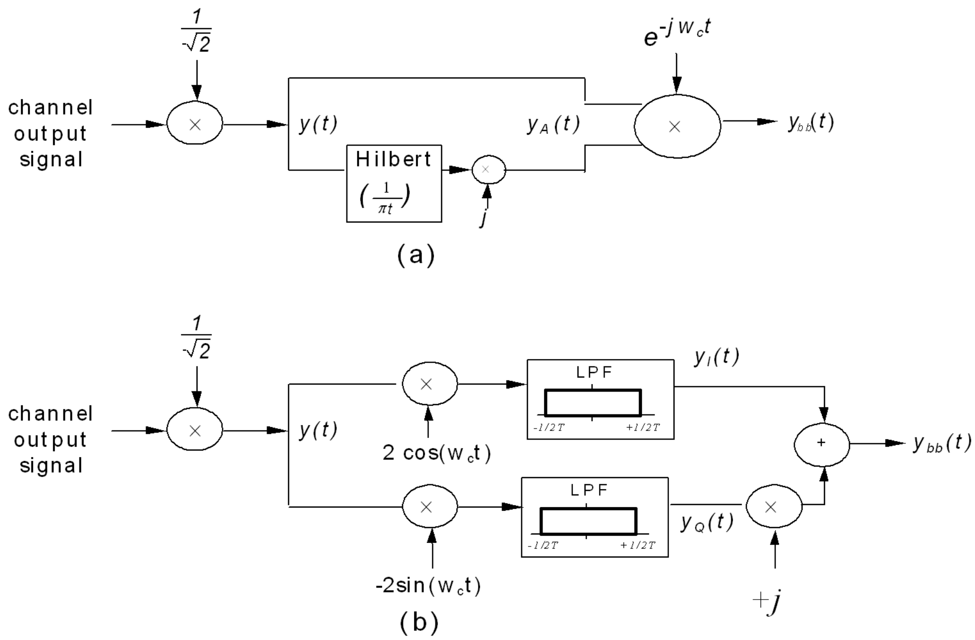

Figure 2.10: Complex demodulator.

which after multiplication by the carrier-demodulating term $e^{-\jmath\omega_c t}$ provides a baseband equivalent signal $y_{bb}(t) = e^{-\jmath\omega_c t} \cdot y_A(t)$

$$y_{bb}(t) = \frac{1-3\jmath}{1000\cdot\sqrt{2}} \cdot 1000 \cdot \text{sinc}(10^6 t) + n_{bb}(t) \quad . \tag{2.66}$$

This baseband signal has a real component of $.001/\sqrt{2}$ and an imaginary component of $-.003/\sqrt{2}$ and the component of noise in each of these dimensions is $\frac{\mathcal{N}_0}{2}$. These are indeed the components that would have been associated with this signal if viewed in Chapter 1 as

$$x(t) = \text{sinc}(10^6 t)\cdot\cos(2\pi 10^7 t) + 3\cdot\text{sinc}(10^6 t)\cdot\sin(2\pi 10^7 t) + n(t) = x_1\cdot\varphi_1(t) + x_2\cdot\varphi_2(t) \tag{2.67}$$

where the **normalized** basis functions are

$$\varphi_1(t) \;=\; \sqrt{2}\cdot 1000\cdot\text{sinc}(10^6 t)\cos(\omega_c t) \tag{2.68}$$

$$\varphi_2(t) \;=\; -\sqrt{2}\cdot 1000\cdot\text{sinc}(10^6 t)\sin(\omega_c t) \quad . \tag{2.69}$$

If the baseband signal $y_{bb}(t)$ is now passed through the matched filter $1000\cdot\text{sinc}(10^6 t)$ and sampled at time 0, the two components $.001/\sqrt{2}$ and $-.003/\sqrt{2}$ are obtained on the real and imaginary dimensions of the output. The relevant independent noise component in each of these dimensions is $\frac{\mathcal{N}_0}{2}$, the power-spectral density of the original white noise, which is equal to its variance per dimension. Thus, this example illustrates how the QAM signal is recovered using the matched filter (if noise is zero)of Chapter 1 exactly with no gain or scaling factors on the original components if the scaling demodulator in Figure 2.10 is also used in the receiver.

The subsequent subsection now adds a filtering channel with impulse response $h(t)$ to the system and investigates modeling of the filtered AWGN as a complex baseband equivalent for QAM transmission.

2.3.2 Generating the baseband equivalent: some examples

The transmission engineer is nominally presented with a variety of information about the transmission channel for which they must design a modem. Often in this author's experience, this information is not

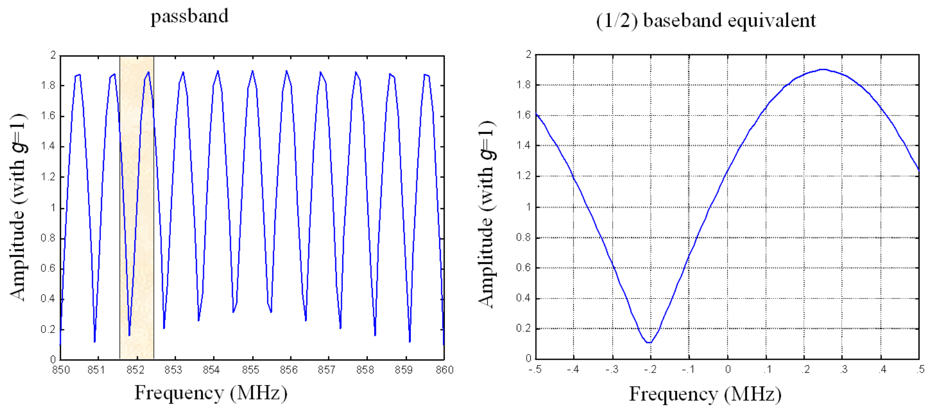

Figure 2.11: Illustration of passband and (1/2 times) baseband-equivalent for wireless two-ray multipath channel.

in a convenient form initially, and a considerable amount of time and effort is spent understanding and transforming the channel. Part of this understanding involves generating the baseband-equivalent channel so that the tools of analysis in this text can be applied. The first example of a two-ray mathematical model for a channel may be easier for the student to follow mathematically because it starts with a very plausible and tractable mathematical model. The second example starts with information that must be converted to an acceptable response.

EXAMPLE 2.3.2 (Two-ray wireless channel) Wireless transmission sometimes is (over-simply) modeled by a two-ray model. A transmission path between transmit and receive antennas has then two paths between the antenna, one direct and one indirect. The latter path usually represents a reflection from a building, mountain, or other physical entity. The second path is usually delayed (say by τ=1.1 μs) and attenuated (say 90% of the amplitude of the first path) with respect to the first path. Let us then say that the channel impulse response is then

$$h(t) = g\left(\delta(t) - .9\delta(t-\tau)\right) \quad , \tag{2.70}$$

where g is an attenuation factor that models the path loss and antenna losses. The Fourier transform is

$$H(f) = g\left(1 - .9e^{-\jmath 2\pi f \tau}\right) \quad . \tag{2.71}$$

The noise will be white and a combination of a number of factors, natural and man-made with one-sided PSD -150 dBm/Hz. Wireless systems often use carrier frequencies between 800 and 900 MHz, so let us choose a carrier for QAM modulation at 852 MHz and further choose 4QAM transmission with a symbol rate of $1/T$=1.0 MHz. Thus, frequencies between 852-.5 = 851.5 MHz and 852+.5=852.5 MHz are of interest, corresponding to a baseband-equivalent channel with frequencies between -500 kHz and + 500 kHz.

The complex-channel model for this transmission is then

$$\frac{1}{2}h_{bb}(t) = g\left(\delta(t) - .9\delta(t-\tau)\right) \cdot e^{-\jmath 2\pi f_c t} \tag{2.72}$$

with Fourier transform

$$\frac{1}{2}H_{bb}(f) = H(f+f_c) = g\left(1 - .9e^{-\jmath 2\pi (f+f_c)\tau}\right) \quad . \tag{2.73}$$

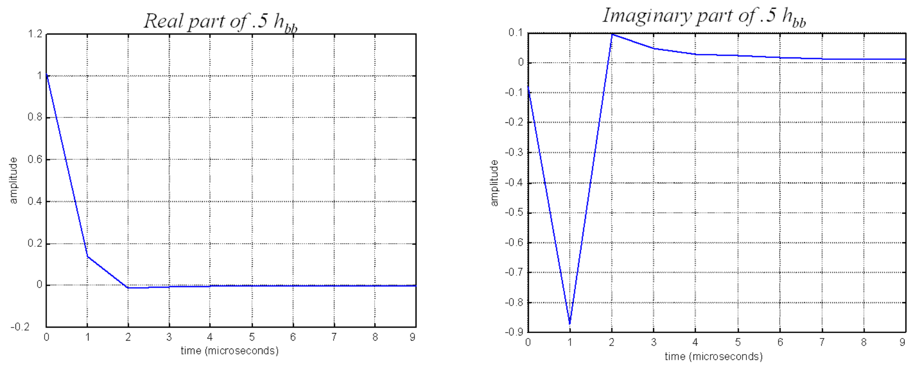

Figure 2.12: Real and imaginary parts of (scaled) baseband equivalent channel response for two-ray example.

Figure 2.11 plots the original channel response from 850 MHz to 860 MHz, along with the complex channel for baseband modeling (with factor of 1/2 included) from -500 kHz to 500 kHz. The channel clearly has "notching" effects because of the possibility of the second path adding out-of-phase (with phase π) at some frequencies. The wider the bandwidth of a QAM transmission system, the more likely one (or more) of the "dips" is to occur in the transmission band of interest. Thus, this "multipath" distortion will lead to a non-flat or filtered-AWGN channel response (which means the techniques of Chapter 3 and later chapters are necessary for reliable recovery of messages). The baseband-equivalent (actually 1/2 amplitude is included in the plot) is clearly not symmetric about frequency zero, meaning its baseband equivalent impulse response is complex, as the formula above in (2.72) also implies. The real and imaginary parts of the baseband-equivalent response appear in Figure 2.12. The baseband-equivalent noise for the model introduced in this Section is still white and has $\mathcal{N}_0$=-150 dBm/Hz, or equivalently $\mathcal{N}_0 = 10^{-18}$. For typical values of g in well designed transmission systems, this will be a few orders of magnitude below the signal levels. It's very simple in this case: Slide the Fourier transform in the band of interest down to DC, then set the complex noise level equal to the single-sided PSD.

In more sophisticated transmission design, the 2-ray model that easily led to a nice compact mathematical description is a rarity. More likely, the engineer will be given (or will have to measure himself) the channel frequency attenuation in dB at several frequencies in the band of transmission, along with the measured delay of signals through the channel at each of these frequencies. The noise is also likely to have been measured within bands centered around each of the measured channel frequencies (or perhaps at other frequencies). The process of conversion to a complex baseband channel may be tedious, but follows the same steps as in the next example.

EXAMPLE 2.3.3 (Telephone Line Channel) Telephone lines today are sometimes used for transmission of data within the home at speeds as high as 10 Mbps, using a transmission system sometimes known as Home Phone Network Access (HPNA) or "G/pnt". The carrier frequency is 7.5 MHz and the symbol rate is 5 MHz for a 4 QAM signal. Telephone line attenuation versus frequency is often measured in terms of "insertion loss" in dB, a ratio of the voltage at the line output to the voltage at the same load point when the phone line is removed. For a well-matched system, it can be determined that this insertion loss is 6 dB above the transfer function from source to load, which is the desired function for digital transmission analysis. Figure 2.13 plots the insertion loss in dB for a 26-gauge phone line of length 300 meters. The baseband equivalent channel response is in the frequency range from 5 MHz to 10 MHz, which the designer "slides" so that 7.5 MHz now appears as DC, as also illustrated in Figure 2.13. Note that the characteristic has been increased by 6 dB to get the transfer function, and again the baseband complex channel shown includes the scale factor

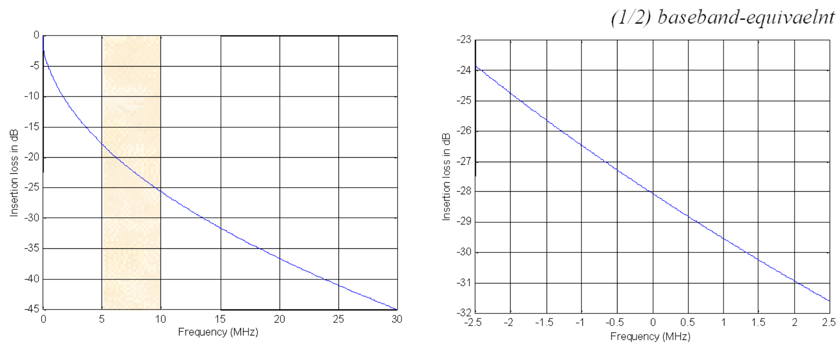

Figure 2.13: Illustration of passband insertion loss and (1/2 times) baseband-equivalent transfer function for home-phone network example.

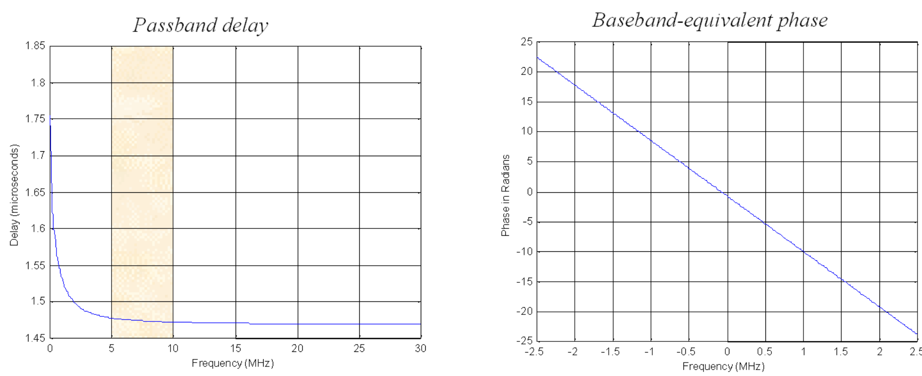

Figure 2.14: Illustration of passband delay and baseband-equivalent phase for home-phone network example.

of 1/2. The designer would presumably be given (or has measured) the insertion loss at a sufficient number of frequencies between 5 and 10 MHz, stored those values in a file, and now is analyzing them with digital signal processing. To use common digital signal processing operations like the inverse Discrete Fourier Transform, the measured values will need to be equally spaced in frequency between 5 MHz and 10 MHz. Let us say here that 501 measurements with spacing 10 kHz have been so taken. These 501 values form the amplitudes (after conversion of dB back into linear-scale values) of the channel transfer function at the frequencies 5 MHz, 5.01 MHz, ... 10 MHz, or for baseband (increased by 6 dB to compute transfer function from insertion loss) equivalent from -2.5 MHz to 2.5 MHz.

The line is also characterized in terms of its delay at all these same frequencies, usually measured in microseconds and plotted in Figure 2.14. The index will be from $n = 0, ..., 500$ across the frequency band of interest. Since delay is negative the derivative of phase, to compute the phase angles for the baseband equivalent transfer function, the phase needs to be accumulated (with minus sign) from -2.5 MHz to each and every frequency of interest according to

$$\angle H_{bb}(-2.5\text{MHz } + n \cdot .01\text{MHz}) = \theta_0 - \sum_{i=0}^{n} \text{Delay}\left[H_{bb}(i)\right] \quad , \tag{2.74}$$

where θ_0 is an constant arbitrary phase reference that ultimately has no effect on transceiver performance, and thus usually taken to be 0. The baseband equivalent channel (scaled by

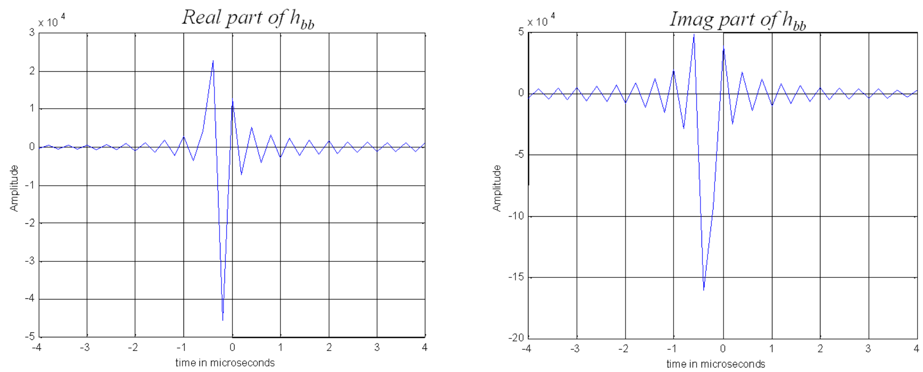

Figure 2.15: Illustration of baseband equivalent complex channel for home-phone network example.

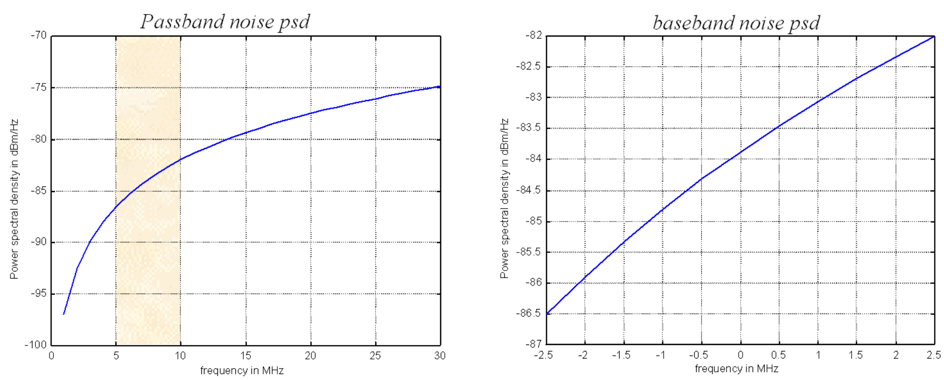

Figure 2.16: Illustration of baseband-equivalent complex noise for home-phone network example.

1/2) is then found by inverse DFT'ing (IFFT command in Matlab) the vector of values $H_{bb}(n)$ $n = 0, ..., 500$. Because of the arbitrary phase, the time-domain response is usually not centered and has nonzero components at the beginning and end of the response. Simple circular shift (already included in Figure 2.14) will provide a "centered" $h_{bb}(t)$ sampled at the symbol rate (which can be made causal by simple reindexing of the time axis). The transmit psd of the 4QAM signal is about -57 dBm/Hz, so that the power is then about 10 dBm (or 10 milliwatts). To interpolate the baseband response to finer time-resolution than the symbol rate, a band wider than 5-10 MHz must be measured, translated to DC, and then inverse transformed.

An interesting effect in telephone-line transmission is that neighbors' data signals can be "heard" through electromagnetic coupling between phone lines in phone cables "upstream" of the home. Thus, the noise is not "white," and a simple model for the one-sided power spectral density of this noise has power spectral density:

$$-187 + 15\log_{10}(f) \ \text{dBm/Hz} \ . \qquad (2.75)$$

This power-spectral density can be computed with f values from 5 MHz to 10 MHz, and then translated to baseband to obtain the baseband-equivalent psd as in Figure 2.16. To find the so-called "white-noise equivalent" in Section 1.7 for this complex baseband equivalent channel, the inverse noise psd can be IFFT'd to the time-domain and factored using the roots command in matlab. Terms with roots of magnitude greater than 1 correspond to the minimum-phase factorization, said inverse can then be convolved with the channel $.5h_{bb}(kT)$ to find the white-noise equivalent channel samples at the symbol rate.

This last example seems like much tedious work, and it is perhaps simple comparied to what communication engineers do. It is provided to emphasize how important it is for communications designers to know and well model their channel so that the theories and guidance learned from this text can be applied. Thus, the excuse "well, nothing I learned had any value" is more likely that the engineer did not understand the level of work involved in making what has been learned of value.

2.3.3 Complex generalization of inner products and analysis

The theory of optimum demodulation for complex signals in the case of the baseband equivalent WGN channel, or generally any complex channel (see later sections), is essentially the same as that for real signals in Chapter 1. All analysis and structure of detectors previously derived carries through with the following generalizations for complex arithmetic:

1. The inner product becomes

$$< \boldsymbol{x}, \boldsymbol{y} >= \boldsymbol{x}^* \boldsymbol{y} = \int_{-\infty}^{\infty} x^*(t) y(t) dt \quad , \tag{2.76}$$

   ($\boldsymbol{x}^*$ means conjugate transpose of $\boldsymbol{x}$).

2. The matched filter is conjugated, that is $\varphi(T-t) \rightarrow \varphi^*(T-t)$.

3. Energies of complex scalars are $\mathcal{E}_{\boldsymbol{x}} = E\left\{|x(t)|^2\right\}$, or the expected magnitude of the complex scalar, and $\bar{\mathcal{E}}_{\boldsymbol{x}} = \mathcal{E}_{\boldsymbol{x}}/2$.

2.4 Passband Analysis for QAM alternatives

Passband analysis directly applies to QAM modulation in a way that simply requires computing the baseband equivalent of a channel for convolution with the complex input $(x_1 + \jmath x_2)\varphi(t)$. Some transmission systems instead may use one of the three other implementations mentioned in Section 1.6, VSB, CAP, or OQAM. This section addresses the specifics of how the passband analysis concepts discussed so far still apply to these other passband modulation types. In all cases, a complex-equivalent channel can be found easily from the given channel transfer function.

2.4.1 Passband VSB Analysis

This section starts with SSB (single-side-band) and then generalizes to VSB. With SSB, the transmitted signal has $x_I(t)$ and $x_Q(t)$ that are Hilbert transforms of one another and thus

$$x(t) = x_I(t) \cdot \cos(\omega_c t) - \check{x}_I(t) \cdot \sin(\omega_c t) \quad . \tag{2.77}$$

Such a signal only exhibits nonzero energy content for frequencies exceeding the carrier frequency (and for frequencies below the negative of the carrier frequency). The baseband equivalent of an SSB signal is also therefore analytic, for which we introduce the new notation

$$x_{Ab}(t) = x_{bb}(t) = x_I(t) + \jmath \check{x}_I(t) \quad . \tag{2.78}$$

The subscript of *Ab* is intended to represent a new signal that is both analytic and baseband and used in SSB analysis. The baseband equivalent of a channel output is consistently

$$y_{Ab}(t) \quad = \quad x_{Ab}(t) * \left(\frac{h_{Ab}(t)}{2}\right) \tag{2.79}$$

$$Y_{Ab}(\omega) \quad = \quad X_{Ab}(\omega) \cdot H(\omega + \omega_c) \;\; \omega > 0 \quad . \tag{2.80}$$

With a little thought, the reader should realize that the previous analysis of passband channels applies for any carrier frequency and not just one centered within the passband. Thus, baseband-equivalent analysis directly applies to SSB also and the “Ab” notation has just made the position of the carrier frequency on the lower edge of the band explicit. However, the input construction is such that $x_Q(t)$ is no longer independent of $x_I(t)$. Generally speaking, with this SSB constraint, twice as many dimensions per second are transmitted within $x_I(t)$ for SSB than would be the case for QAM with the same bandwidth. However, QAM can independently use the quadrature dimension whereas this dimension is completely determined from the inphase dimension with SSB. The analysis for lower sideband (instead of the assumed upper sideband in this analysis) follows by simply negating the quadrature component and choosing the carrier frequency at the upper edge of the passband, then the baseband equivalent is nonzero for negative frequencies only, but still a special case of the general analysis of Sections 2.1 and 2.3.

VSB transmission is based on SSB transmission. In general, VSB systems have $x_I(t)$ and $x_Q(t)$ selected in such a way that they are almost Hilbert transforms of one another. A VSB system may be easier to implement in practice and is always based on an equivalent SSB signal. With the nomenclature of this text, the baseband equivalent of a VSB signal, for which we introduce the notation $x_{Vb}(t)$ has “vestigial” symmetry about $f = 0$ – that is $X_{Vb}(f) + X_{Vb}(-f) = X_{Ab}(f) \; \forall \; f > 0$ where $X_{Ab}(f)$ is for the (analytic) SSB signal in (2.78) upon which the VSB signal is based. The VSB signal is based on a carrier frequency that is not at the edge of the passband. This carrier frequency is the point around which the passband signal exhibits vestigial symmetry. This frequency is again selected for the baseband equivalent representation of the channel,

$$Y_{Vb}(\omega) = X_{Vb}(\omega) \cdot H(\omega + \omega_c) \;\; \omega > -\omega_c \quad . \tag{2.81}$$

Terrestrial digital television broadcast in the USA uses VSB transmission with carrier frequencies at the nominal TV carrier positions of 52 MHz + i·(6 MHz), effectively $\bar{b} = 2$ (constellation is coded so it is called 64 VSB, where extra levels are redundant for coding, see Chapter 10) and a symbol rate of roughly 5 MHz, for a data rate of 20 Mbps. The signals thus have non-zero energy from about 1.5 MHz below the carrier and to 3.5 MHz above the carrier using vestigial transmit symmetry with respect to that carrier.

2.4.2 Passband CAP Analysis

Analysis of CAP (carrierless amplitude phase) modulation essentially relies on analytic signal and channel equivalents instead of baseband equivalents.

A CAP signal is generated according the following observation of Werner for the sum (sequence) of analytic QAM signals (x_k can be complex and represents the two-dimensional QAM symbol transmitted at symbol time instant k and $\varphi(t)$ is the baseband equivalent modulating function)

$$\begin{aligned} x_A(t) &= \sum_k x_k \varphi(t-kT) e^{\jmath\omega_c t} && (2.82) \\ &= \sum_k x_k \varphi(t-kT) e^{\jmath\omega_c t} \cdot e^{-\jmath\omega_c kT} \cdot e^{+\jmath\omega_c kT} && (2.83) \\ &= \sum_k (x_k \cdot e^{+\jmath\omega_c kT}) \varphi(t-kT) \cdot e^{\jmath\omega_c (t-kT)} && (2.84) \\ &= \sum_k \tilde{x}_k \varphi_A(t-kT) && (2.85) \end{aligned}$$

where the new quantities are defined as

$$\varphi_A(t) = \varphi(t) \cdot e^{\jmath\omega_c t}$$

and $\tilde{x}_k = x_k \cdot e^{+\jmath\omega_c kT}$ (a rotated version of the input).[5] Thus, a CAP system uses simple rotation of the encoder outputs to create a symbol-time-invariant realization of the subsequent modulation. Since the sequence of rotations is known, the receiver need only detect $\tilde{x}_k$, and x_k can easily be determined by reverse rotations,

$$x_k = \tilde{x}_k e^{-\jmath\omega_c kT} \quad .$$

In practice, the rotations are ignored and the sequence $\tilde{x}_k$ itself directly carries the information, noting that the rotations at each end simply undo each other and have no bearing on performance or functionality. They are necessary only for equivalence to a QAM signal.

The channel output of interest is then the analytic channel output so that

$$\begin{aligned} y_A(t) &= x_A(t) * \left(\frac{h_A(t)}{2} \right) && (2.86) \\ Y_A(\omega) &= X_A(\omega) \cdot H(\omega) \;\; \omega > 0 \quad . && (2.87) \end{aligned}$$

With CAP then, only the analytic signals are of interest, and the complex channel that describes the method is the analytic equivalent channel that is found by zeroing the Fourier transform for negative frequencies (and for which we introduce the notation H_{CAP} specific to analysis of CAP transmission over a channel with response generally noted by $h(t)$.):

$$H_{CAP}(\omega) = H(\omega) \cdot \frac{1}{2} \left(1 + sgn(\omega)\right) \quad . \tag{2.88}$$

In practice, on a channel with narrow transmission band relative to the carrier frequency, intermediate-frequency (IF) demodulation is used to reduce (but not zero) the effective center frequency of the transmission band closer to DC. Then CAP is applied to the IF demodulated signal. The IF demodulation treats the transmission signals as if they were analog signals and can be considered outside the realm and interest of digital data transmission.

Nonetheless, after conversion to complex equivalent channels, both QAM and CAP receiver processing can be generally described by the processing of a complex channel output, and such a complex model is the objective of this chapter.

[5] This $\tilde{x}_k$ is notation used specific to the CAP situation here, and is not intended to be equivalent to any other temporary uses of a tilde on a quantity elsewhere in this textbook.

2.4.3 OQAM or "Staggered QAM"

The basis functions for OQAM appear in Section 1.6 of Chapter 1. One could rewrite this entire Chapter with $\cos(\omega_c t)$ replaced by $\text{sinc}(t/T)\cos(\omega_c t)$ and most importantly $\sin(\omega_c t)$ replaced by $\text{sinc}([t - T/2]/T)\sin(\omega_c t)$ everywhere – and all results would still hold. However, there is an easier way to reuse what has already been derived: Now, with the reader's understanding of passband signals, a transmission engineer can say that the essential difference between OQAM and QAM is that the quadrature component is delayed by one-half symbol period with respect to the inphase component in OQAM. The designer can analyze a new equivalent channel input with double the symbol rate and a time-varying encoder that alternates between a nonzero inphase component (with zero quadrature component) and a nonzero quadrature component (with zero inphase component. This new time-varying double-speed symbol sequence can then be applied to a conventional QAM modulator to generate the OQAM sequence. All results developed so far then apply to this new equivalent system running at twice the symbol rate. The energy per dimension $\bar{\mathcal{E}}_{\boldsymbol{x}}$ will reduce by a factor of 2 if power is maintained constant.

Thus, to find the output of a channel with impulse response $h(t) = \Re\{h_{bb}(t)e^{\jmath\omega_c t}\}$ to an OQAM input, simply convolve the baseband equivalent of the continuous-time $x(t)$ formed from the double-symbol-rate "interleaved" symbol sequence with $[h_{bb}(t)]/2$. Again a complex channel will have been constructed for analysis – the objective for this chapter.

The inphase and quadrature dimensions are not strictly independent since they have alternating zero values. This dependence or correlation effectively halves the bandwidth so that an OQAM system running with symbol rate $1/T$ and the basis functions in Section 1.6, even though analyzed as a QAM system running with interdependent symbols at rate $2/T$, occupies the same bandwidth as QAM. In fact as the function $\varphi_i(t)$ is generalized (See Chapter 3) so that $\varphi(t) \neq \sqrt{1/T}\text{sinc}(t/T)$, then OQAM typically requires less bandwidth in terms of the inevitable "non-brick-wall" energy roll-off associated with practical filter design.

Exercises - Chapter 2

2.1 *Passband Representations.*

Consider the following passband waveform:

$$x(t) = sinc^2(t)(1 + A\sin(4\pi t))\cos(\omega_c t + \frac{\pi}{4}),$$

where $\omega_c >> 4\pi$.

Hint: It may be convenient in working this problem to use the identity $cos(a+b) = \cos a \cos b - \sin a \sin b$ to rewrite $x(t)$ in inphase and quadrature, and to realize/define $sinc^2(t)$ equal to a more general pulse shaping function $p(t)$, recognizing that this particular choice of $p(t)$ has a fourier transform that is well known and easily sketched.

a. Sketch (roughly) $\mathrm{Re}[X(\omega)]$ and $\mathrm{Im}[X(\omega)]$. (2 pts)

b. Find $x_{bb}(t)$, the baseband equivalent of $x(t)$. Sketch (roughly) $X_{bb}(\omega)$. (3 pts)

c. Find the $x_A(t)$ analytic equivalent of $x(t)$. (2 pts)

d. Find the Hilbert Transform of $x(t)$. (2 pts)

2.2 *A two tap channel.*

The two equally likely baseband signals,$\tilde{x}_{bb,1}(t)$ and $\tilde{x}_{bb,2}(t)$ illustrated in the following figure are used to transmit a binary sequence over a channel. The use of the scaling phase splitter in Figure 2.8 is assumed. Note that the two signals do not seem to be of the form $(x_1 + \jmath x_2)\varphi(t)$ directly. However, this form can be applied if one views each of these two signals as a succession of four "one-shot" inputs to the channel, each of which can be construed as of the form $(x_1 + \jmath x_2)\varphi(t - iT/4)$, $i = 0, 1, 2, 3$ – this view is not necessary, however, to work this problem. Equivalently, an easy representation is a four-dimensional symbol vector.

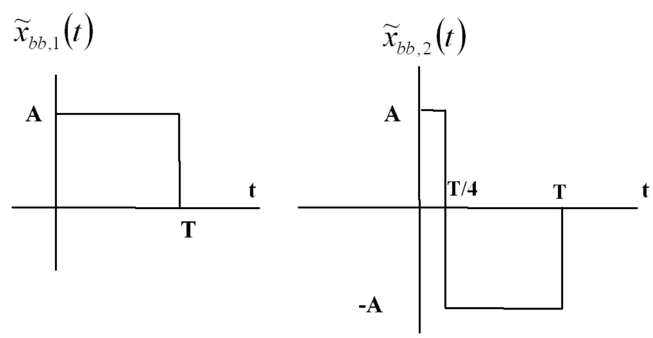

Figure 2.17: Two baseband signals.

The baseband equivalent channel impulse response is

$$h_{bb}(t) = 4\delta(t) - 2\delta(t - T)$$

The transmission rate is $R = \frac{1}{2T}$ bits per second to avoid pulse overlay.

a. Sketch the two possible baseband equivalent noise–free received waveforms. (2 pts)

b. Compute the squared distance between the two baseband possible signals as the integral of the squared difference between the two signals at the channel input. Compute the same distance after filtering by the channel impulse response.(2 pts)

c. Determine P_e for transmission of the two corresponding messages where $n_{bb}(t)$ has autocorrelation $r_{bb}(t) = \mathcal{N}_0\delta(t)$ with $\mathcal{N}_0 = \frac{1}{20}\mathcal{E}_x$ where $\mathcal{E}_x$ is the average energy before filtering by the channel response. (4 pts)

2.3 *A bandpass channel.* (from **Proakis**)
The input $x(t)$ to a bandpass filter is

$$x(t) = u(t)cos(w_c t)$$

where

$$u(t) = \begin{cases} A & \text{if } t \in [0, T] \\ 0 & \text{otherwise} \end{cases}$$

Please assume that w_c is sufficiently high that $x(t)$ has only a neglible amount of energy near DC.

a. (3 pts) Determine the output $y(t) = g(t) * x(t)$ of a bandpass filter for all $t \geq 0$ if the impulse response of the filter is,

$$g(t) = \begin{cases} \frac{2}{T}e^{-t/T}\cos(\omega_c t) & \text{if } t \in [0, T] \\ 0 & \text{otherwise} \end{cases}$$

b. Sketch the equivalent lowpass output of the filter if it is passed through the scaling phase-splitter, $\tilde{y}_{bb}(t) =?$ (1 pt)

c. Assume the baseband equivalent noise at the output of the scaling phase splitter has variance $\mathcal{N}_0$ and that there are two dimensions. What is the SNR of the output? (3 pts)

d. For what value of A is the SNR$_{channel\ output}$=13 dB if the power spectral density of the channel's AWGN ($\frac{\mathcal{N}_0}{2}$) is - 30 dBm/Hz and $1/T = 1000$ Hz? Repeat for -100 dBm/hz and 1 MHz respectively.

2.4 *Passband equivalent system.*
A baseband-equivalent waveform ($w_c > 2\pi$)

$$\tilde{x}_{bb}(t) = (x_1 + jx_2)\text{sinc(t)}$$

is convolved with the complex filter

$$w_1(t) = \delta(t) - j\delta(t-1)$$

a. (1 pt) Find

$$y(t) = w_1(t) * \tilde{x}_{bb}(t).$$

b. (1 pt) Suppose y(t) is convolved with the imaginary filter

$$w_2(t) = 2j\text{sinc(t)}$$

to get

$$\begin{aligned} z(t) &= w_2(t) * y(t) \\ &= w_2(t) * w_1(t) * \tilde{x}_{bb}(t) \\ &= w(t) * \tilde{x}_{bb}(t). \end{aligned}$$

Find $z(t)$. Note that $sinc(t) * sinc(t-k) = sinc(t-k)$, k an integer.

c. (3 pts) Let

$$\tilde{z}(t) = Re\left(z(t)e^{jw_c t}\right) = \tilde{w}(t) * x(t) \quad ,$$

where $x(t) = \Re\{\tilde{x}_{bb}(t)e^{\jmath\omega_c t}\}$. Show that

$$\tilde{w}(t) = 4sinc(t-1)cos(w_c t) - 4sinc(t)sin(w_c t)$$

when convolved with the passband $x(t)$ will produce $\tilde{z}(t)$. Hint: Use baseband calculations.

2.5 *Matlab demodulator.*

For this problem you will need three matlab files: **x.mat, plt_fft.m,** and **plt_cplx.m**. You can get these files from the class web-site. If absolutely necessary, the TA can copy the necessary files onto a floppy provided by you or email you the files.

In this problem you will demodulate three symbols of a passband 4 QAM signal. Our baseband basis function is a windowed sinc function. The sampling rate that provided our digital received signal was 1000 Hz. As is typical of such systems, the received signal that you will start with is the real part of the analytic signal. Usually the signal will have been convolved with a channel response and had noise added, but we have been blessed with an extremely clean channel.

a. Obtain the three needed files. Execute **load x.mat** which will create a 747 point vector x which contains 3 symbol periods of received signal. (Each symbol period is 249 samples). The function **plt_cplx** plots the real and imaginary parts of a complex vector. It takes two arguments, the vector and the plot title. Execute **plt_cplx(x, 'received signal')** and turn in your plot. Note that the received signal is indeed real. While you cannot identify the constellation points, you should be able to easily spot the three symbols. (1 pt)

b. Now execute **plt_fft(x, 'received signal')** to plot an FFT of the received signal. Turn in your plot. Neglecting powers below -50 dB, to what range of frequencies is this signal bandlimited? (2 pts)

c. Now use the function **hilbert()** provided by MATLAB to recover the analytic signal **x_A**. You might want to execute **help hilbert** to get started. Use **plt_fft** to plot the FFT of **x_A**. Turn in your plot. How is this signal different from **x**? In your discussion, you may again neglect signal power below -50 dB. (2 pts)

d. The carrier frequency is 250 Hz. As mentioned before, our sampling frequency is 1000 Hz. Show that the discrete time radian carrier frequency is $\frac{\pi}{2}$ radians/sec. (1 pt)

e. (4 pts) Now demodulate **x_A** to create the baseband signal **x_bb** by executing the following command:

x_bb = x_A .* exp (-j * 0.5 * pi * [0:746]);

Plot both the FFT and complex time sequence as before, and turn in your plots. In what range of frequencies is **x_bb** non-negligible? Why? By examining the complex time sequence plot, decode the received signal. The complex constellation points have been labeled as shown below. To select the correct constellation point for each symbol, you really need only consider the sign of the real and imaginary scale factors multiplying the windowed sinc pulse.

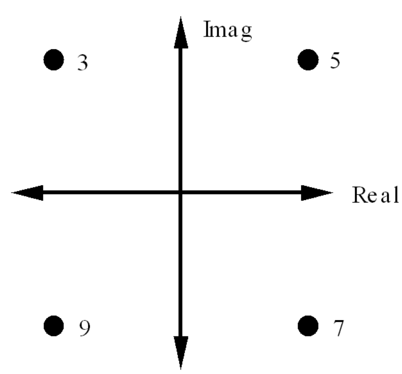

Figure 2.18: 4-QAM constellation with message labels 3, 5, 7 & 9

2.6 *Baseband Analysis*

Consider the two baseband equivalent signals, $\tilde{x}_{bb,1}(t)$ and $\tilde{x}_{bb,2}(t)$.

$$\tilde{x}_{bb,1}(t) = \begin{cases} A\,(1+j) & \text{if } t \in [0,T] \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{x}_{bb,2}(t) = \begin{cases} A\,(1-j) & \text{if } t \in \left[0, \frac{3}{4}T\right] \\ -A\,(1-j) & \text{if } t \in \left(\frac{3}{4}T, T\right] \\ 0 & \text{otherwise} \end{cases}$$

$$j = \sqrt{-1}$$

These signals can be used to transmit a binary signal set.

$$\mathrm{p}_{\boldsymbol{x}_{bb}}(1) = \mathrm{p}_{\boldsymbol{x}_{bb}}(2) = \frac{1}{2}$$

The transmitted signals are corrupted by AWGN having a baseband equivalent representation corresponding to the scaled phase splitter of Figure 2.8, $\tilde{n}_{bb}(t)$, with an autocorrelation function

$$r_{\tilde{n}_{bb}}(\tau) = E\left[\tilde{n}^*_{bb}(t)\tilde{n}_{bb}(t+\tau)\right] = \mathcal{N}_0\delta(t)$$

a. Find $\mathcal{E}_{\boldsymbol{x}}$. (2 pts)

b. Find P_e as a function of A, T and $\mathcal{N}_0$. (1 pts)

c. Find $\frac{A^2}{N_0}$ in terms of T if SNR=12.5 dB. Compute the probability of error P_e (a numerical value is required). (1 pts)

2.7 *Twisted pairs*

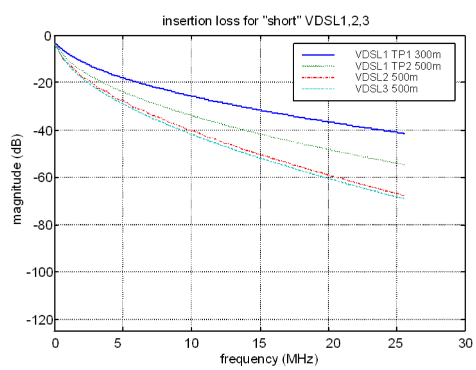

Figure 2.19: Insertion loss for 26-gauge (TP1) and 24-guage (TP2) twisted-pair phone line.

Figure 2.19 shows the magnitude(in dB) of the insertion losses (which is 6 dB more than the transfer function $H(f)$) for several lengths of two-types of twisted pair. Suppose the signals are passband, with frequencies ranging from 6MHz to 12 MHz. You want to analyse this transmission system in baseband. Assume a carrier frequency of $f = 9$ MHz. Assume the receiver uses a scaling phase splitter.

a. Draw the frequency responses of the complex channels to which you would apply the complex modulator input $\tilde{x}_{bb}(t)$, corresponding to the scaling in Figure 2.8. (1 pt)

b. Compute the noise power spectral density (two-sided) of the WGN that you would add to each of your complex channel outputs to model transmission if the one-sided power spectral density of the AWGN noise on the channel is given as -140 dBm/Hz. (2 pts)

2.8 *Signal transformation practice*
Find the Hilbert transform of:

$$x(t) = sinc(\frac{t}{T}) \cos(\omega_c t + \frac{\pi}{4})$$

where

$$\omega_c \geq \frac{\pi}{T} \quad .$$

2.9 *baseband analysis - Midterm 1994*
The two baseband equivalent signals at the modulator output using the scaling in Figure 2.8 for binary transmission:

$$\begin{aligned}
\tilde{x}_0(t) &= \frac{1}{\sqrt{T}}\text{sinc}\left(\frac{t}{T}\right) + \jmath\sqrt{\frac{2}{T}}\cos\left(\frac{\pi t}{T}\right) \cdot \text{sinc}(t/T) \\
\tilde{x}_1(t) &= \jmath\sqrt{\frac{2}{T}}\sin\left(\frac{\pi t}{T}\right) \cdot \text{sinc}(t/T)
\end{aligned}$$

are transmitted over an AWGN with $\frac{\mathcal{N}_0}{2} = .02$. Determine P_e.

2.10 *Complex channel evaluation - Midterm 1995*
A passband channel has complex baseband equivalent impulse response

$$h_{bb}(t) = (1 + \jmath)\delta(t) \quad .$$

A 4 QAM (QPSK) input with the constellation labeling below in Figure 2.20 is input to this channel. WGN is added at the output of this channel with power spectral density $\frac{\mathcal{N}_0}{2} = .04$.

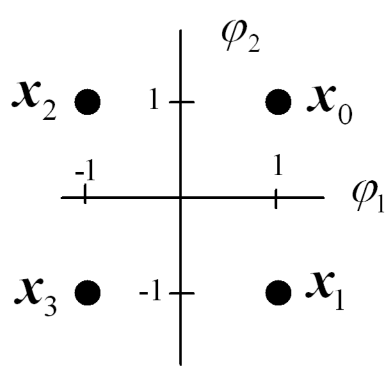

Figure 2.20: Channel input for Problem 2.10

a. Draw the complex signal constellation points x_{bb}. (1 pt)

b. Calculate and draw the constellation for the corresponding four signal constellation points at the output of this channel and a scaling demodulator. Call these points y_0, ..., y_3 where the subscripts correspond to the subscripts on the channel input. (2 pts)

c. Write a quadrature decomposition for y_0's corresponding passband modulated signal. (2 pts)

d. Calculate and draw the constellation for the sampled matched-filter outputs in the absence of noise. (3 pts)

e. Sketch the baseband AWGN power spectral density. (2 pts)

2.11 *64 VSB - Midterm 1996*

Let $m(t)$ be an 8-PAM signal with

$$m(t) = \sum_k x_k p(t - kT)$$

where $x_k = \pm 1, \pm 3, \pm 5, \pm 7$ and $p(t) = \text{sinc}(2t/T)$. Also let $\omega_c >> 2\pi/T$ while forming the modulated signal

$$x(t) = m(t)\cos(\omega_c t) - \check{m}(t)\sin(\omega_c t)$$

for transmission over an AWGN.

a. Find the Hilbert transform of $p(t)$.

b. Write an expression for $\check{m}(t)$ and evaluate this expresssion in terms of integer multiple of $T/2$ sampling instants.

c. If $x_k = \delta_k$ for this part (c) only, and find $x_{bb}(t)$ and sketch $X_{bb}(f)$ and $X_A(f)$. δ_k is defined as the value 1 for $k = 0$ and zero for any other integer k.

d. Draw the block diagram of an ML receiver.

e. Describe the action of the detector.

f. Why is this signal called **64**-VSB or equivalently **64**-SSB?

g. For what SNR is $P_e < 10^{-6}$?

2.12 *Complex Channel - Midterm 1997*

A complex channel, derived through the scaling of Figure 2.8, has binary inputs $\tilde{x}_{bb}(t)$ in Figure 2.21 below. Let the passband filter be $\frac{h_{bb}(t)}{2} = \delta(t)$ and the SNR $= \frac{\bar{\mathcal{E}}_{\boldsymbol{x}}}{\sigma^2}$=10 dB and the signal is considered two dimensional (one real and one imaginary dimension) for computation of $\bar{\mathcal{E}}_{\boldsymbol{x}}$.

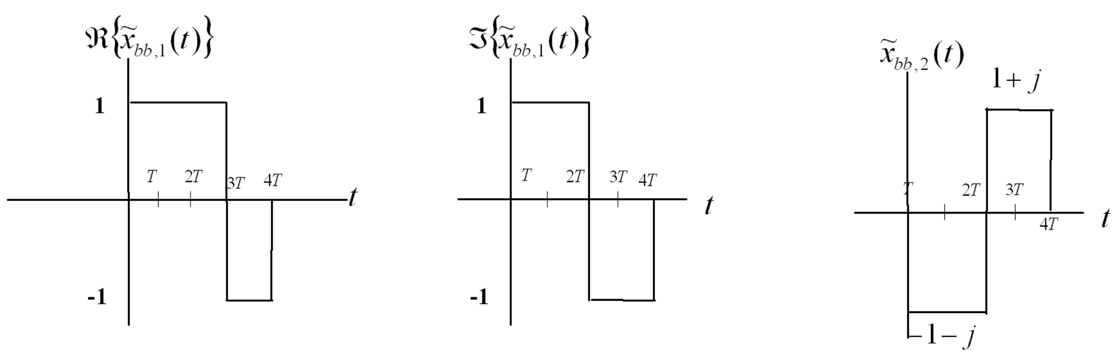

Figure 2.21: Baseband channel inputs.

a. write $\tilde{x}_{bb,1}(t)$ and $\tilde{x}_{bb,2}(t)$ in the form of 4 successive transmissions with symbol rate $1/T$. Each symbol must be of the form $(x_1 + \jmath x_2) \cdot \varphi(t - kT)$ $k = 0, 1, 2, 3$. Thus, you must find x_1, x_2, and φ. (4pts)

b. Find d_{min} with $T = 1$. (2 pts)

c. Find P_e for an ML detector with $T = 1$. (2 pts)

d. For the remainder of this problem, let $\frac{1}{2}h_{bb}(t) = \delta(t) + \delta(t - 1)$ and $T = 1$. Find the baseband-equivalents channel outputs prior to addition of baseband noise, $\tilde{n}_{bb}(t)$. (4 pts)

e. Find d_{min}. (2 pts)

f. Find P_e with an ML detector, assuming all inputs are equally likely. (2 pts)

g. Has the distortion introduced by the new $h_{bb}(t)$ improved or degraded this system? Why? (3 pts)

2.13 *Baseband Equivalents - Midterm 1998*

The figure below shows the Fourier transform, $H(f)$, of a bandlimited channel's impulse response, $h(t)$. The input to the channel is

$$x(t) = \sqrt{2}\left\{\left[\sum_k a_k \cdot \varphi(t-kT)\right] \cdot \cos(\omega_c t) - \left[\sum_k b_k \cdot \varphi(t-kT)\right] \cdot \sin(\omega_c t)\right\} \quad . \tag{2.89}$$

This channel is used for passband transmission with QAM and has AWGN with (2-sided) power spectral density $\sigma^2 = .01$.

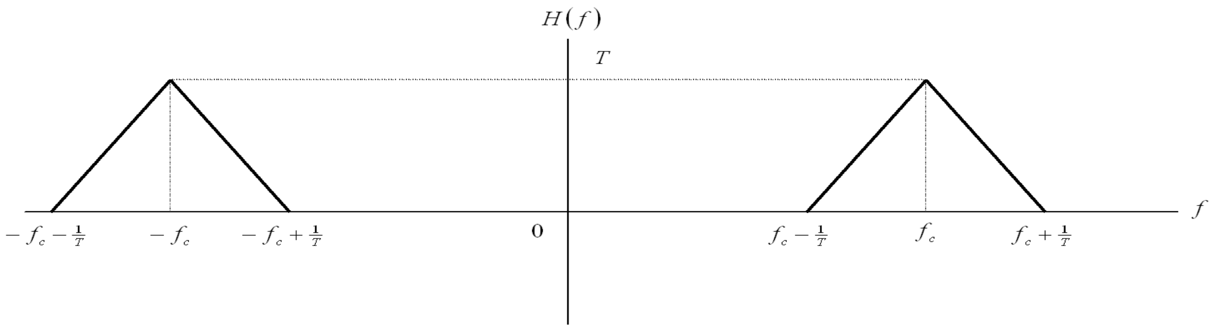

Figure 2.22: Figure for "baseband equivalents".

a. Draw $H_{bb}(f)$. (1 pt)

b. Find $h_{bb}(t)$. (2 pts)

c. Find the input $\tilde{x}_{bb}(t)$ as per Figure 2.8 in the class text. (2 pts)

d. Find the complex channel model, including the channel complex impulse response and a numerical value for the noise power spectral density corresponding to the complex input you found in part c. The channel output should be the $\tilde{y}_{bb}(t)$ of Figure 2.8 in the class text. (2 pts)

e. Find the analytic equivalent $h_A(t)$. (2 pts)

f. Write a simple expression for the Hilbert transform of $h(t)$. $\check{h}(t) =$?(2 pts)

2.14 *Baseband Channel - Midterm 2000 - 10 pts*

The Fourier transform of the impulse response of a channel is shown in the Figure below. The power spectral density of the additive Gaussian noise at the output of the channel is shown in the second figure below. You are given the following integrals to avoid any need for doing integration in this problem (i.e., you can plug the formulae)

$$\int x^2 e^{bx} dx = \frac{e^{bx}}{b^3}\left(b^2x^2 - 2bx + 2\right) \;\; ; \;\; \int x e^{bx} dx = \frac{e^{bx}}{b^2}(bx-1) \;\; ; \;\; \int e^{bx} dx = \frac{e^{bx}}{b} \quad . \tag{2.90}$$

Throughout this problem, please use the scaling QAM demodulator of Figure 2.10.

a. Draw the Fourier transform of the complex channel, $\frac{1}{2}H_{bb}(f)$, that is used in our class to model the channel. (2 pts)

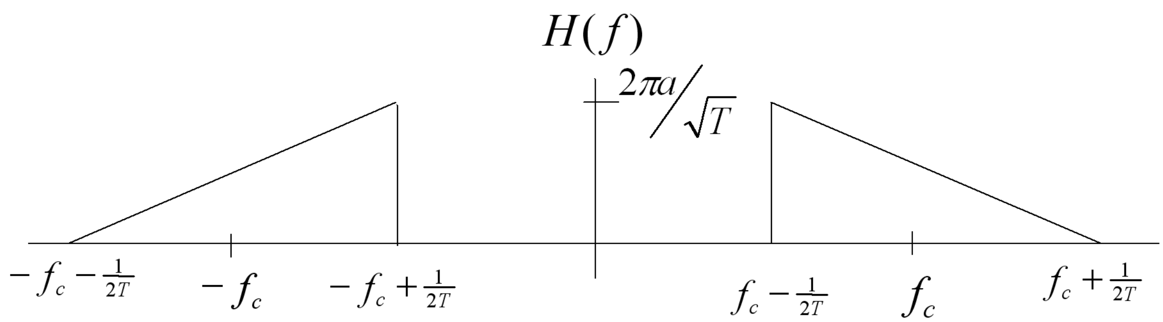

Figure 2.23: Channel Response.

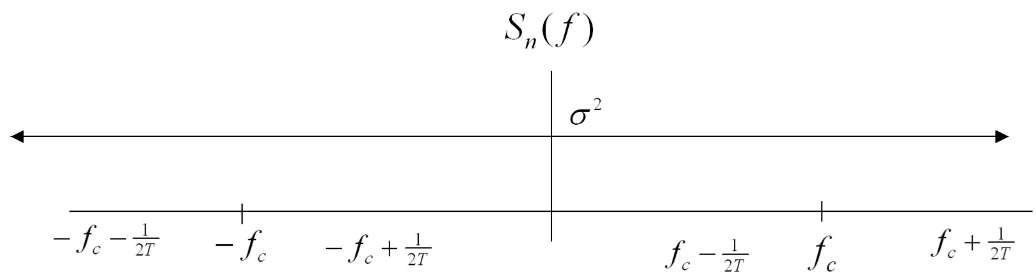

Figure 2.24: Noise Power Spectral Density.

b. Find the power spectral density per real dimension of the noise in the complex baseband-equivalent channel that results from the scaled demodulator. (1 pt)

c. Find the complex-equivalent pulse response (time-domain) of the channel in part a if the transmitter uses the basis functions of QAM with $\varphi(t) = \frac{1}{\sqrt{T}} \cdot \mathrm{sinc}\frac{t}{T}$. (4 pts)

d. If $a = \sqrt{T}$ and $\sigma^2 = 1$, and the transmitter sends 16 SQ QAM with constellation points $\begin{bmatrix} \pm 3 \\ \pm 1 \end{bmatrix}$, what is the lowest upper bound on the best possible SNR for a symbol-by-symbol detector on this channel? (3 pts)

2.15 *Mini-Design - Midterm 2001 - 12 pts*

An AWGN with SNR=22 dB has baseband channel transfer function (there is no energy gain or loss in the channel):

$$H(f) = \begin{cases} 1 & |f| < 500 \text{ kHz} \\ 0 & |f| \geq 500 \text{ kHz} \end{cases} \tag{2.91}$$

a. Find an integer $\bar{b}$ for QAM transmission with $P_e < 10^{-6}$ and compare with value found by the "gap approximation." (2 pts)

b. Find the largest possible symbol rate and corresponding data rate. (2 pts)

c. Draw the corresponding signal constellation and label your points with bits. (2 pts) - hint, don't concern yourself with clever bit labelings, just do it.

d. Find $\bar{P}_b$ and N_b for your design in part b. (3 pts)

e. Repart part b for PAM transmission on this channel and explain the difference in data rates. (3 pts)

2.16 *Baseband Channel 2 - Midterm 2001 - 12 pts*

For the AWGN channel with transfer function shown in Figure 2.25, a transmitted signal cannot exceed 1 mW (0 dBm) and the power spectral density is also limited according to $S_x(f) \leq$-83 dBm/Hz(two-sided psd). The two-sided noise power spectral density is σ^2 =-98 dBm/Hz. The carrier frequency is f_c =100 MHz for QAM transmission. The probability of error is $P_e = 10^{-6}$.

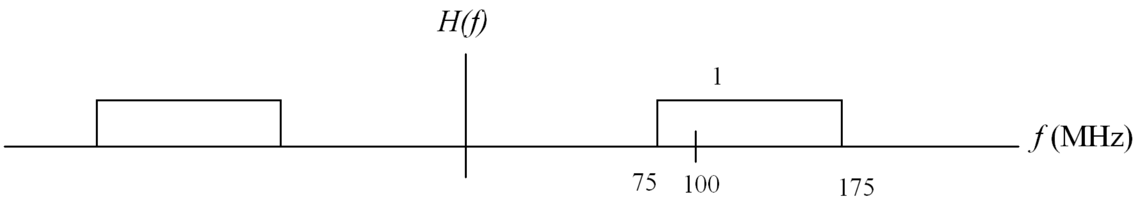

Figure 2.25: Channel Response.

a. Find the baseband channel model, $\frac{1}{2}H_{bb}(f)$, for the scaled demodulator of Chapter 2. (2 pts)

b. Find the largest symbol rate that can be used with the 100 MHz carrier frequency? (1 pts)

c. What is the maximum signal power at the channel output with QAM? (2 pts)

d. What QAM data rate can be achieved with the symbol rate of part b? (2 pts)

e. Change the carrier frequency to a value that allows the best QAM data rate. (2 pts)

f. What is the new data rate for your answer in part e? (3 pts)

2.17 *Complex Channel and Design- Midterm 2003 - 15 pts*

Let $x_k$represent the successive independent transmitted symbols of a QAM constellation that can have only an integer number of bits for each symbol and for which each message is equally likely. Also let the pulse response of a filtered AWGN channel be $p(t) = \sqrt{\frac{1}{T}} \cdot \text{sinc}\left(\frac{t}{T}\right)$ where $\frac{1}{T} = 5$ Mhz is the symbol rate of the QAM transmission. The carrier frequency is 100 MHz. The transmitted signal has $\bar{\mathcal{E}}_{\boldsymbol{x}} = 1.2$ and the AWGN psd is $\sigma^2 = .01$. An expression for the modulated signal is

$$x(t) = \Re\left\{\sum_k x_k \cdot p(t-kT) \cdot e^{\jmath\omega_c t}\right\} \quad . \tag{2.92}$$

Define $g(t) = p(t) \cdot e^{\jmath\omega_c t}$.

a. Find $x_A(t)$ and $x_{bb}(t)$. (2 pts)

b. Use the gap approximation to determine the number of bits per dimension, data rate, and the number of bits/second/Hz that can be transmitted on this channel with $\bar{P}_e \leq 10^{-6}$. (3 pts)

c. Suppose $A_k = x_k \cdot e^{\jmath\omega_c kT}$ is the actual message symbol sequence of interest for the rest of this problem. How do your answers in part b change (if at all)? Why? (2 pts)

d. Find the analytic equivalent of the channel output, $y_A(t)$ in terms of only the message sequence A_k and $g(t)$ without any direct use of the carrier frequency. (2 pts)

e. Draw an optimum (MAP) detector for A_k that does not use the carrier frequency directly. (3 pts)

f. Augment your answer in part e with a simple rotation that provides the MAP detector for x_k. (1 pt)

g. This approach is used in some communications systems where the symbol rate and carrier frequency can be co-generated from the same oscillator, hence the knowledge of the symbol rate in the receiver tacitly implies then also knowing the carrier frequency. This is why the carrier was eliminated in the receiver of part e. Suggest a receiver implementation problem with this approach in general to replace QAM systems that would be used in transmission with symbol rates of up to 10 MHz and carriers above 1 GHz. (Hint - what does $g(t)$ look like?) (2 pts)

2.18 *Complex Channel and Modulation - Midterm 2005 - 15 pts*

A 16 QAM constellation is used to transmit a message over a filtered AWGN with SNR $= \frac{\bar{\mathcal{E}}_{\boldsymbol{x}}}{\sigma^2} = 20$ dB. The QAM symbol is given by $\sqrt{\frac{2}{T}} \cdot \left[x_1 \cdot f(t) + x_2 \check{f}(t)\right]$ where $f(t) = \text{sinc}\left(\frac{2t}{T}\right)$. The real channel filter in has the shape shown in Figure 2.26 and is zero outside the frequency band of $(-1/T, 1/T)$.

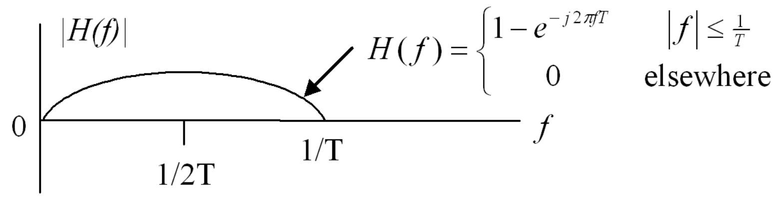

Figure 2.26: Channel Response.

a. Show $f(t) = \text{sinc}\left(\frac{t}{T}\right) \cdot \cos(\pi t)$ using frequency-domain arguments. Then find the Hilbert Transform $\check{f}(t)$. (2 pts)

b. Find a quadrature representation for $x(t)$ and the appropriate carrier frequency f_c. (2 pts)

c. Find $x_{bb}(t)$ and $x_A(t)$. (2 pts)

d. Find $h(t)$, $h_{bb}(t)$, and $h_A(t)$. **Hint:** multiplication of a transfer function by the ideal lowpass filter is the same as convolving with a sinc function, which may be useful in converting the obvious frequency-domain answers to the time domain. (2 pts)

e. For the scaled phase splitter of Chapter 2, find the output $y_{bb}(t)$. (2 pts)

f. Design (draw) an optimum receiver for this single transmitted message using only one sampling device, one matched filter, and one adder. Draw the decision regions for a single complex value that ultimately emanates from your receiver. (3 pts)

g. Calculate P_e for this optimum receiver. (2 pts)

Appendix A

Hilbert Transform

The Hilbert transform is a linear operator that shifts the phase of a sinusoid by 90^o:

Definition A.0.1 (Hilbert Transform) *The Hilbert Transform of $x(t)$ is denoted $\check{x}(t)$ and is given by*

$$\check{x}(t) = \hbar(t) * x(t) = \int_{-\infty}^{\infty} \frac{x(u)}{\pi(t-u)} du \quad , \tag{A.1}$$

where

$$\hbar(t) = \begin{cases} \frac{1}{\pi t} & t \neq 0 \\ 0 & t = 0 \end{cases} \quad . \tag{A.2}$$

The Fourier Transform of $\hbar(t)$ is

$$\begin{aligned}
\mathcal{H}(\omega) &= \int_{-\infty}^{\infty} \frac{e^{-\jmath\omega t}}{\pi t} dt & \text{(A.3)} \\
&= \int_{-\infty}^{\infty} \frac{-\jmath \sin \omega t}{\pi t} dt & \text{(A.4)} \\
&= \int_{-\infty}^{\infty} \frac{-\jmath \sin \pi u}{\pi u} \mathrm{sgn}(\omega) du & \text{(A.5)} \\
&= -\jmath \left[\int_{-\infty}^{\infty} \mathrm{sinc}(u) du \right] \mathrm{sgn}(\omega) & \text{(A.6)} \\
&= -\jmath \mathrm{sgn}(\omega) & \text{(A.7)}
\end{aligned}$$

Equation (A.7) shows that a frequency component at a positive frequency is shifted in phase by -90^o, while a component at a negative frequency is shifted by $+90^o$. Summarizing

$$\check{X}(\omega) = -\jmath \mathrm{sgn}(\omega) X(\omega) \quad . \tag{A.8}$$

Since $|\mathcal{H}(\omega)| = 1 \; \forall \; \omega \neq 0$, then $|X(\omega)| = |\check{X}(\omega)|$, assuming $X(0) = 0$. This text only considers passband signals with no energy present at DC ($\omega = 0$). Thus, the Hilbert Transform only affects the phase and not the magnitude of a passband signal.

A.1 Examples

Let

$$x(t) = \cos(\omega_c t) = \frac{1}{2}(e^{\jmath\omega_c t} + e^{-\jmath\omega_c t}) \quad , \tag{A.9}$$

then

$$\check{x}(t) = \frac{1}{2}(-\jmath e^{\jmath\omega_c t} + \jmath e^{-\jmath\omega_c t}) = \frac{1}{2\jmath}(e^{\jmath\omega_c t} - e^{-\jmath\omega_c t}) = \sin(\omega_c t) \quad . \tag{A.10}$$

Let

$$x(t) = \sin(\omega_c t) = \frac{1}{2\jmath}(e^{\jmath\omega_c t} - e^{-\jmath\omega_c t}) \quad , \tag{A.11}$$

then

$$\check{x}(t) = \frac{1}{2j}(-\jmath e^{\jmath\omega_c t} - \jmath e^{-\jmath\omega_c t}) = -\frac{1}{2}(e^{\jmath\omega_c t} + e^{-\jmath\omega_c t}) = -\cos(\omega_c t) \quad . \tag{A.12}$$

Note

$$\check{\check{x}}(t) = \hbar(t) * \hbar(t) * x(t) = -x(t) \quad . \tag{A.13}$$

since $(-\jmath\mathrm{sgn}(\omega))^2 = -1 \,\forall\, \omega \neq 0$. A correct interpretation of the Hilbert transform is that every sinusoial component is passed with the same amplitude, but with its phase reduced by 90 degrees.

A.2 Inverse Hilbert

The inverse Hilbert Transform is easily specified in the frequency domain as

$$\mathcal{H}^{-1}(\omega) = \jmath\mathrm{sgn}(\omega) \quad , \tag{A.14}$$

or then

$$\hbar^{-1}(t) = -\hbar(t) = \begin{cases} -\frac{1}{\pi t} & t \neq 0 \\ 0 & t = 0 \end{cases} \quad . \tag{A.15}$$

A.3 Hilbert Transform of Passband Signals

Given a passband signal $x(t)$, form the quadrature decomposition

$$x(t) = x_I(t)\cos(\omega_c t) - x_Q(t)\sin(\omega_c t) \tag{A.16}$$

and transform $x(t)$ into the frequency domain

$$X(\omega) = \frac{1}{2}\left[X_I(\omega+\omega_c) + X_I(\omega-\omega_c)\right] - \frac{1}{2\jmath}\left[X_Q(\omega-\omega_c) - X_Q(\omega+\omega_c)\right] \quad . \tag{A.17}$$

Equation A.17 shows that if $X(\omega) = 0 \;\forall\; |\omega| > 2\omega_c$ then $X_I(\omega) = 0$ and $X_Q(\omega) = 0 \;\forall\; |\omega| > \omega_c$.[1] Using this fact the Hilbert transform $\check{X}(\omega)$ is given by

$$\check{X}(\omega) = \frac{\jmath}{2}\left[X_I(\omega+\omega_c) - X_I(\omega-\omega_c)\right] + \frac{1}{2}\left[X_Q(\omega-\omega_c) + X_Q(\omega+\omega_c)\right] \; . \tag{A.18}$$

The inverse Fourier Transform of $\check{X}(\omega)$ then yields

$$\check{x}(t) = x_I(t)\sin(\omega_c t) + x_Q(t)\cos(\omega_c t) = \Im\{x_A(t)\} \;, \tag{A.19}$$

where $\Im$ denotes the imaginary part.

[1] Recall $x_I(t)$ and $x_Q(t)$ are real signals.

Appendix B

Passband Processes

This appendix investigates properties of the correlation functions for a WSS passband random process $x(t)$ in its several representations. For a brief introduction to the definitions of random processes see Appendix C.

B.1 Hilbert Transform

Let $x(t)$ be a WSS real-valued random process and $\check{x}(t) = \hbar(t) * x(t)$ be its Hilbert transform. By Equation (C.8), the autocorrelation of $\check{x}_t$ is

$$r_{\check{x}}(\tau) = \hbar(\tau) * \hbar^*(-\tau) * r_x(\tau) = r_x(\tau). \tag{B.1}$$

Since $|\mathcal{H}(\omega)|^2 = 1 \quad \forall\, \omega \neq 0$, and assuming $S_X(0) = 0$, then

$$S_{\check{x}}(\omega) = S_x(\omega) \quad . \tag{B.2}$$

Thus, a WSS random process and its Hilbert Transform have the same autocorrelation function and the same power spectral density.

By Equation (C.13)

$$r_{\check{x},x}(\tau) = \hbar(\tau) * r_x(\tau) = \check{r}_x(\tau) \tag{B.3}$$

$$r_{x,\check{x}}(\tau) = \hbar^*(-\tau) * r_x(\tau) = -\hbar(\tau) * r_x(\tau) = -\check{r}_x(\tau) \quad . \tag{B.4}$$

The cross correlation between the random process $x(t)$ and its Hilbert transform $\check{x}(t)$ is the Hilbert transform of the autocorrelation function of the random process $x(t)$.

Note also that

$$r_{\check{x},x}(\tau) = \hbar(\tau) * r_x(\tau) = \hbar^*(\tau) * r_x(\tau) = \hbar^*(\tau) * r_x(-\tau) = r^*_{x,\check{x}}(-\tau) = r_{x,\check{x}}(-\tau) \quad . \tag{B.5}$$

By using Equations (B.3), (B.4) and (B.5),

$$r_{\check{x},x}(\tau) = \check{r}_x(\tau) = -r_{x,\check{x}}(\tau) = -r_{\check{x},x}(-\tau) \quad . \tag{B.6}$$

Equation (B.6) implies that $r_{\check{x},x}(\tau)$ is an odd function, and thus

$$r_{\check{x},x}(0) = 0\ . \tag{B.7}$$

That is, a real-valued random process and its Hilbert Transform are uncorrelated at any particular point in time.

B.2 Quadrature Decomposition

The quadrature decomposition for any real-valued WSS passband random process and its Hilbert transform is

$$\begin{aligned} x(t) &= x_I(t)\cos\omega_c t - x_Q(t)\sin\omega_c t && \text{(B.8)} \\ \check{x}(t) &= x_I(t)\sin\omega_c t + x_Q(t)\cos\omega_c t. && \text{(B.9)} \end{aligned}$$

The baseband equivalent complex-valued random process is

$$x_{bb}(t) = x_I(t) + \jmath x_Q(t) \tag{B.10}$$

and the analytic equivalent complex-valued random process is

$$x_A(t) = x(t) + \jmath\check{x}(t) = x_{bb}(t)e^{\jmath\omega_c t} \ . \tag{B.11}$$

The original random process can be recovered as

$$x(t) = \Re\left\{x_A(t)\right\} \quad . \tag{B.12}$$

The autocorrelation of $x_A(t)$ is

$$\begin{aligned} r_A(\tau) &= E\left\{x_A(t)x_A^*(t-\tau)\right\} && \text{(B.13)} \\ &= 2\left(r_x(\tau) + \jmath\check{r}_x(\tau)\right) \quad . && \text{(B.14)} \end{aligned}$$

The right hand side of Equation (B.14) is twice the analytic equivalent of the autocorrelation function $r_x(\tau)$. The power spectral density is

$$S_A(\omega) = 4 \cdot S_x(\omega) \quad \omega > 0 \ \ . \tag{B.15}$$

The functions in the quadrature decomposition of $x(t)$ also have autocorrelation functions:

$$\begin{aligned} r_I(\tau) &\triangleq E\left\{x_I(t)x_I^*(t-\tau)\right\} && \text{(B.16)} \\ r_Q(\tau) &\triangleq E\left\{x_Q(t)x_Q^*(t-\tau)\right\} && \text{(B.17)} \\ r_{IQ}(\tau) &\triangleq E\left\{x_I(t)x_Q^*(t-\tau)\right\} && \text{(B.18)} \end{aligned}$$

One determines

$$\begin{aligned} r_x(\tau) &= E\left\{x(t)x^*(t-\tau)\right\} && \text{(B.19)} \\ &= r_I(\tau)\cos\omega_c t\cos\omega_c(t-\tau) \\ &- r_{IQ}(\tau)\cos\omega_c t\sin\omega_c(t-\tau) \\ &- r_{QI}(\tau)\sin\omega_c t\cos\omega_c(t-\tau) \\ &+ r_Q(\tau)\sin\omega_c t\sin\omega_c(t-\tau) \end{aligned}$$

Standard trigonometric identities simplify (B.19) to

$$\begin{aligned} r_x(\tau) = \ & \frac{1}{2}\left[r_I(\tau) + r_Q(\tau)\right]\cos\omega_c\tau && \text{(B.20)} \\ & +\frac{1}{2}\left[r_{IQ}(\tau) - r_{QI}(\tau)\right]\sin\omega_c\tau \\ & -\frac{1}{2}\left[r_Q(\tau) - r_I(\tau)\right]\cos\omega_c(2t-\tau) \\ & -\frac{1}{2}\left[r_{IQ}(\tau) + r_{QI}(\tau)\right]\sin\omega_c(2t-\tau) \end{aligned}$$

Strictly speaking, most modulated waveforms are WS cyclostationary with period $T_c = \frac{2\pi}{\omega_c}$, i.e. $E\left[x(t)x^*(t-\tau)\right] = r_x(t, t-\tau) = r_x(t+T_c, t+T_c-\tau)$ (See Appendix C). For cyclostationary random processes a time-averaged autocorrelation function of one variable τ can be defined by $r_x(\tau) = 1/T_c \int_{-T_c/2}^{T_c/2} r_x(t, t-\tau)dt$, and this new time-averaged autocorrelation function will satisfy the properties derived thus far in this section. The next set of properties require that the random process to be WSS, not WS cyclostationary – or equivalently the time-averaged autocorrelation function $r_x(\tau) = 1/T_c \int_{-T_c/2}^{T_c/2} r_x(t, t-\tau)dt$ can be used. An example of a WSS random process is AWGN. Modulated signals often have equal energy inphase and quadrature components with the inphase and quadrature signals derived independently from the incoming bit stream; thus, the modulated signal is then WSS.

For $x(t)$ to be WSS, the last two terms in (B.20) must equal zero. Thus, $r_I(\tau) = r_Q(\tau)$ and $r_{IQ}(\tau) = -r_{QI}(\tau) = -r_{IQ}(-\tau)$. The latter equality shows that $r_{IQ}(\tau)$ is an odd function of τ and thus $r_{IQ}(0) = 0$. For $x(t)$ to be WSS, the inphase and quadrature components of $x(t)$ have the same autocorrelation and are uncorrelated at any particular instant in time. Substituting back into Equation B.20,

$$r_x(\tau) = r_I(\tau)\cos(\omega_c\tau) - r_{QI}(\tau)\sin(\omega_c\tau) \tag{B.21}$$

Equation B.21 expresses the autocorrelation $r_x(\tau)$ in a quadrature decomposition and thus

$$\check{r}_x(\tau) = r_I(\tau)\sin(\omega_c\tau) + r_{QI}(\tau)\cos(\omega_c\tau) \tag{B.22}$$

Further algebra leads to

$$\begin{aligned} r_{bb}(\tau) &= E\left\{x_{bb}(t)x_{bb}^*(t-\tau)\right\} && \text{(B.23)} \\ &= 2\left(r_I(\tau) + \jmath r_{QI}(\tau)\right) && \text{(B.24)} \\ r_A(\tau) &= r_{bb}(\tau)e^{\jmath\omega_c\tau} && \text{(B.25)} \end{aligned}$$

The power spectral density is

$$S_A(\omega) = S_{bb}(\omega - \omega_c) \quad . \tag{B.26}$$

If $\mathcal{S}_x(\omega)$ is symmetric about ω_c, then $S_{bb}(\omega)$ is symmetric about $\omega = 0$ (recall that the spectrum $S_A(\omega)$ is a scaled version of the positive frequencies of $S_X(\omega)$ and $S_{bb}(\omega)$ is $S_A(\omega)$ shifted down by ω_c). In this case $r_{bb}(\tau)$ is real, and using Equation B.24, $r_{QI}(\tau) = 0$. Equivalently, the inphase and quadrature components of a random process are uncorrelated at any lag τ (not just $\tau = 0$) if the power spectral density is symmetric about the carrier frequency. Finally,

$$r_x(\tau) = \frac{1}{2}\Re\left\{r_{bb}(\tau)e^{\jmath\omega_c\tau}\right\} = \frac{1}{2}\Re\left\{r_A(\tau)\right\} \quad . \tag{B.27}$$

If $x(t)$ is a random modulated waveform, by construction it is usually true that $r_I(\tau) = r_Q(\tau)$ and $r_{IQ}(\tau) = -r_{QI}(\tau) = 0$, so that the constructed $x(t)$ is WSS. For AWGN, $n(t)$ is usually WSS so that $r_I(\tau) = r_Q(\tau)$ and $r_{IQ}(\tau) = -r_{QI}(\tau) = 0$. When a QAM waveform is such that $r_{IQ}(\tau) \neq -r_{QI}(\tau)$ or $r_I(\tau) \neq r_Q(\tau)$, then $x(t)$ is WS cyclostationary with period π/ω_c.

Appendix C

Random Processes (By Dr. James Aslanis)

This appendix briefly outlines the fundamental definitions of random processes used in this textbook. Random variables and processes are specified by upper-case letters in this appendix, and the corresponding lower case letters are used as variables of integration. Other sections and appendices relax notation and use the lower-case notation for both.

The natural mathematical construct to describe a noisy communication signal is a random process. Random processes are simply a generalization of random variables. Consider a single sample of a random process, which is described by a random variable X with probability density $P_X(x)$. (Often the random process is assumed to be Gaussian but not always). This random variable only characterizes the random process for a single instant of time. For a finite set of time instants, a random vector $\boldsymbol{X} = [X_{t_1}, \ldots, X_{t_n}]$ with a joint probability density function $P_{\boldsymbol{X}}(\boldsymbol{x})$ describes the noise. Extension to a countably infinite set of random variables, indexed by i, defines a discrete-time random sequence.

> **Definition C.0.1 (Discrete-Time Random Sequence)** *A Discrete-Time Random Sequence $\{X_i\}$ is a countably infinite, indexed set of random variables described by a joint density function $P_{\boldsymbol{X}}(\boldsymbol{x})$ where $\boldsymbol{X} = [X_{t_i}, \ldots, X_{t_{i+n}}]$, where i is an integer and n is a positive integer.*

The random variables in a random process need not be independent nor identically distributed, although the random variables are all defined on the same sample space. The probability density functions that define a random process do depend on the indices. If the index i indicates time, then the probability densities are time-varying. Similarly, statistical averages, i.e. $E[f(X_t)]$, become functions of the index as well. These statistical averages, also known as ensemble averages, should not be confused with averaging over the time index (sometimes referred to as time averaging). For example the mean value of a random variable associated with the tossing of a die is approximated by averaging the values over many independent tosses. The time average is said to converge to the ensemble average (a property also referred to as **mean ergodic**). In a random process, each random variable in the collection of random variables may have a different probability density function. Time averaging over successive samples may not yield any information about ensemble averages.

In a communication system, we often observe a realization (function of time) of the random process, also called a sample function. For repetition of the experiment, a different sample function is observed. Thus, the random process X_t also can be considered an ensemble of sample functions. Statistical averages are computed over the ensemble of sample functions. For certain processes the time averages (i.e. averaging over a single sample function the sequential values in time) may approximate well the ensemble averages.

Random processes are classified by statistical properties that their density functions obey, the most important of which is stationarity.

C.1 Stationarity

Definition C.1.1 (Strict Sense Stationarity) *A random process X_t is called* **strict sense stationary (SSS)** *if the joint probability density function*

$$P_{X_{t_1},\ldots,X_{t_n}}(x_{t_1},\ldots,x_{t_n}) = P_{X_{t_1+t},\ldots,X_{t_n+t}}(x_{t_1+t},\ldots,x_{t_n+t}) \; \forall \, n, t, \{t_1,\ldots,t_n\} \quad . \tag{C.1}$$

Roughly speaking the statistics of X_t are invariant to a time shift; i.e. the placement of the origin $t = 0$ is irrelevant.

This text next considers commonly calculated functions of a random process: These functions encapsulate properties of the random processes, and for linear systems we can sometimes calculate these functions for processes without knowing the exact statistics of the process. Certain random processes, such as stationary Gaussian processes, are completely described by a collection of these functions.

Definition C.1.2 (Mean) *The mean of a random process X_t is*

$$E(X_t) = \int_{-\infty}^{\infty} x_t P_{X_t}(x_t) dx_t \triangleq \mu_X(t) \quad . \tag{C.2}$$

In general, the mean is a function of the index t.

Definition C.1.3 (Autocorrelation) *The autocorrelation of a random process X_t is*

$$E(X_{t_1} X_{t_2}^*) = \int_{-\infty}^{\infty} x_{t_1} x_{t_2}^* P_{X_{t_1} X_{t_2}}(x_{t_1}, x_{t_2}) dx_{t_1} dx_{t_2} \triangleq r_X(t_1, t_2) \tag{C.3}$$

In general, the autocorrelation is a two-dimensional function of the pair $\{t_1, t_2\}$. For a stationary process, the autocorrelation is a one-dimensional function of the time difference $t_1 - t_2$ only:

$$E(X_{t_1} X_{t_2}^*) = r_X(t_1 - t_2) = r_X(\tau) \tag{C.4}$$

The stationary process' autocorrelation also satisfies a Hermitian property $r_X(\tau) = r_X^*(-\tau)$.

Using the mean and autocorrelation functions, also known as the first- and second-order statistics, engineers often define a weaker form of stationarity.

Definition C.1.4 (Wide Sense Stationarity) *A random process X_t is called* **wide sense stationary (WSS)** *if*

a. $E(X_t) =$ *constant,*

b. $E(X_{t_1} X_{t_2}^*) = r_X(t_1 - t_2)$*, i.e. a function of the time difference only.*

One can show that SSS $\Rightarrow$ WSS, but WSS $\not\Rightarrow$ SSS. Often, analysis of random processes only considers their first and second order statistics. Such results of course do not reveal anything about the higher order statistics of the random process; however, in one special case the stationary random process is completely defined by the lower order statistics. In particular,

Definition C.1.5 (Gaussian Random Process) *The joint probability density function of a stationary real* **Gaussian random process** *for any set of n indices $\{t_1,\ldots,t_n\}$ is*

$$P_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}(\det \Lambda)^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})\Lambda^{-1}(\boldsymbol{x}-\boldsymbol{\mu})'\right] \quad , \tag{C.5}$$

where the mean vector is given by $\boldsymbol{\mu} = E[\boldsymbol{X}]$, and the covariance matrix is defined as $\Lambda = E[(\boldsymbol{X}-\boldsymbol{\mu})'(\boldsymbol{X}-\boldsymbol{\mu})]$.

A complex Gaussian random variable has independent Gaussian random variables in both the real and imaginary parts, both with the same variance, which is half the variance of the complex random variable. Then, the distribution is

$$P_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(\pi)^n(\det \Lambda)} \exp\left[-(\boldsymbol{x}-\boldsymbol{\mu})\Lambda^{-1}(\boldsymbol{x}-\boldsymbol{\mu})'\right] \quad . \tag{C.6}$$

For a Gaussian random process, the set of random variables $\{X_{t_1}, \ldots, X_{t_n}\}$ are jointly Gaussian. A Gaussian random process also satisfies the following two important properties:

a. The output response of a linear time-invariant system to a Gaussian input is also a Gaussian random process.

b. A WSS, real-valued, Gaussian random process is SSS.

Much of the analysis in this textbook will consider Gaussian random processes passed through linear time-invariant systems. As a result of the properties listed above, the designer only requires the mean and autocorrelation functions of these processes to characterize them completely. Fortunately, one can calculate the effect of linear time-invariant systems on these functions without explicitly using the probability densities of the random process.

In particular, for a linear time-invariant system defined by an impulse response $h(t)$, the mean of the output random process Y_t is

$$\mu_Y(t) = \mu_X(t) * h(t) \quad . \tag{C.7}$$

The autocorrelation of the output can also be found as

$$r_Y(t+\tau, t) = r_X(t+\tau, t) * h(t+\tau) * h^*(-t) \quad . \tag{C.8}$$

In addition, many analyses use the correlation between the input and output random processes:

Definition C.1.6 (Cross-correlation) *The* **cross-correlation** *between the random processes X_t and Y_t is given by*

$$E(X_{t_1} Y_{t_2}^*) \triangleq r_{XY}(t_1, t_2) \quad . \tag{C.9}$$

For a jointly WSS random processes, the cross-correlation only depends on the time difference

$$r_{XY}(t_1, t_2) = r_{XY}(t_1 - t_2) = r_{XY}(\tau) \quad . \tag{C.10}$$

Note that $r_{XY}(\tau)$ does not satisfy the Hermitian property that the autocorrelation obeys, but one can show that

$$r_{XY}(\tau) = r_{YX}^*(-\tau) \quad . \tag{C.11}$$

As an exercise, the reader should verify that

$$r_{XY}(\tau) = r_X(\tau) * h^*(-\tau) \tag{C.12}$$

$$r_{YX}(\tau) = r_X(\tau) * h(\tau) \tag{C.13}$$

A more general form of stationarity is **cyclostationarity**, wherein the statistics of the random process are invariant only to specific shifts in the indices.

Definition C.1.7 (Strict Sense Cyclostationarity) *A random process is* **strict sense cyclostationary** *if the joint probability density function satisfies*

$$P_{X_{t_1},\ldots,X_{t_n}}(x_{t_1}, \ldots, x_{t_n}) = P_{X_{t_1+T},\ldots,X_{t_n+T}}(x_{t_1+T}, \ldots, x_{t_n+T}) \; \forall \; n, \{t_1, \ldots, t_n\}, \tag{C.14}$$

where T is called the **period** *of the process.*

That is, X_{t+kT} is statistically equivalent to $X_t \;\forall\; t, k$. Cyclostationarity accounts for the regularity in communication transmissions that repeat a particular operation at specific time intervals; however, within a particular time interval, the statistics are allowed to vary arbitrarily. As with stationarity, a weaker form for cyclostationarity that depends only on the first and second order statistics of the random process is.

Definition C.1.8 (Wide Sense Cyclostationarity) *A random process is* **wide sense cyclostationary** *if* *a.* $E(X_t) = E(X_{t+kT}) \; \forall \, t, k$.

b. $r_X(t+\tau, t) = r_X(t+\tau+kT, t+kT) \; \forall \, t, \tau, k$.

Thus, the mean and autocorrelation functions of a WS cyclostationary process are periodic functions with period T. Many of the random signals in communications, such as an ensemble of modulated waveforms, satisfy the WS cyclostationarity properties.

The periodicity of a WS cyclostationary random process would complicate the study of modulated signals without use of the following convenient property. Given a WS cyclostationary random process X_t with period T, the random process $X_{t+\theta}$ is WSS if θ is a uniform random variable over the interval $[0, T]$. Thus, we often shall include (or assume) a random phase θ to yield a WSS random process.

Alternatively for a WS cyclostationary random process, we can define a time average autocorrelation function.

Definition C.1.9 (Time Average Autocorrelation) *The* **time average autocorrelation** *of a WS cyclostationary random process* X_t *is*

$$\bar{r}_X(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} r_X(t+\tau, t) dt \quad . \tag{C.15}$$

Since the autocorrelation function $r_X(t+\tau, t)$ is periodic, one could integrate over any closed interval of length T in the preceding equation.

As in the study of deterministic signals and systems, frequency domain descriptions are often useful for analyzing random processes. First, this appendix continues with the definitions for deterministic signals.

Definition C.1.10 (Energy Spectral Density) *The* **Energy Spectral Density** *of a finite energy deterministic signal* $x(t)$ *is* $|X(\omega)|^2$ *where*

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \stackrel{\Delta}{=} \mathcal{F}\{x(t)\} \tag{C.16}$$

is the Fourier transform of $x(t)$*. Thus, the energy is calculable as*

$$\mathcal{E}_x \stackrel{\Delta}{=} \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega < \infty \tag{C.17}$$

If the finite energy signal $x(t)$ is nonzero for only a finite time interval, say T, then the time average power in the signal equals $P_x = \mathcal{E}_x / T$.

Communication signals are usually modeled as repeated patterns extending from $(-\infty, \infty)$, in which case the energy is infinite, although the time average power may be finite.

Definition C.1.11 (Power Spectral Density) *The* **power spectral density** *of a finite power signal defined as*

$$S_x(\omega) = \lim_{T \to \infty} \frac{|X_T(\omega)|^2}{T} \tag{C.18}$$

where $X_T(\omega) = \mathcal{F}(x_T(t))$ *is the Fourier transform of the truncated signal*

$$x_T(t) = \begin{array}{cc} x(t) & |t| < \frac{T}{2} \\ 0 & \text{otherwise} \end{array} \tag{C.19}$$

Thus, the time average power is calculable as $P_x \stackrel{\Delta}{=} \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \int_{-\infty}^{\infty} S_x(\omega) d\omega < \infty$

For deterministic signals, the power P_x is a time-averaged quantity.

For a random process, we cannot simply take its Fourier transform to specify the power spectral density. Even if the integral of the random process is well-defined (and we do not present the requisite mathematical tools in this text), the result would be another random process. Ensemble averages are required for frequency-domain analysis.

For a random process X_t, the ensemble average power, $P_{X_t} \triangleq E[|X_t|^2]$, may vary instantaneously over time. For a WSS random process, however, $P_{X_t} = P_X$ is a constant.

Definition C.1.12 (Power Spectral Density) *For a WSS continuous-time random process X_t, the* **power spectral density** *is*

$$S_X(\omega) = \mathcal{F}\{r_x(\tau)\}. \tag{C.20}$$

One can show that $\int_{-\infty}^{\infty} S_X(\omega)d\omega = P_X$.

For a WS cyclostationary random process X_t the autocorrelation function $r_X(t+\tau, t)$, for a fixed time lag τ, is periodic in t with period T. Consequently the autocorrelation function can be expanded using a Fourier series.

$$r_X(t+\tau, t) = \sum_{n=-\infty}^{\infty} \gamma_n(\tau) e^{j2\pi nt/T}, \tag{C.21}$$

where $\gamma_n(\tau)$ are the Fourier coefficients

$$\gamma_n(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} r_X(t+\tau, t) e^{-j2\pi nt/T} dt. \tag{C.22}$$

The time average autocorrelation function is then

$$\bar{r}_X(\tau) = \gamma_0(\tau). \tag{C.23}$$

The average power for the WS cyclostationary random process is

$$P_{X_t} = \frac{1}{T} \int_{-T/2}^{T/2} E[|X_t|^2]dt = \bar{r}_X(0) = \gamma_0(0) = \int_{-\infty}^{\infty} G_0(f)df \tag{C.24}$$

where $G_0(f)$ is the Fourier transform of the $n = 0$ Fourier coefficient

$$\mathcal{F}\{\gamma_0(\tau)\} = G_0(f) = \mathcal{F}\{\bar{r}_X(\tau)\}. \tag{C.25}$$

The function $G_0(f)$ is the power spectral density of the WS cyclostationary random process X_t associated with the time average autocorrelation $\bar{r}_X(\tau)$.

For a nonstationary random process, the average power must be calculated by both time and ensemble averaging, i.e. $P_{X_t} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} E[|X_t|]^2 dt$.

C.2 Linear Systems

For the linear system defined by $y(t) = h(t) * x(t)$, with a stationary input $x(t)$ and fixed deterministic impulse response $h(t)$, the following relationships are easily proved:

$$r_{YY}(\tau) = h(\tau) * h^*(-\tau) * r_{XX}(\tau) \tag{C.26}$$

and thus

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) \quad . \tag{C.27}$$

These same relations hold in discrete time also.